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Edge-disjoint packing of stars and cycles ☆

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ABSTRACT

We study the parameterized complexity of two graph packing problems, Edge-Disjoint k-Packing of s-Stars and Edge-Disjoint k-Packing of s-Cycles. With respect to the choice of parameters, we show that although the two problems are FPT with both k and s as parameters, they are unlikely to be fixed-parameter tractable when parameterized by only k or only s. In terms of kernelization complexity, we show that Edge-Disjoint k-Packing of s-Stars has a kernel with size polynomial in both k and s, but in contrast, unless NP $\subseteq$ coNP/poly, Edge-Disjoint k-Packing of s-Cycles does not have a kernel with size polynomial in both k and s, and moreover does not have a kernel with size polynomial in s for any fixed k. We also show that Edge-Disjoint k-Packing of 4-Cycles admits a $96k^2$ kernel in general graphs and a $96k$ kernel in planar graphs.

1. Introduction

Given two (simple, undirected) input graphs G and H, the edge-disjoint graph packing problem aims at finding in G the maximum number of edge-disjoint subgraphs isomorphic to H. Denote by *s-star* the complete bipartite graph $K_{1,s}$, and by *s-cycle* the simple cycle graph C_s of length s. In this paper, we study edge-disjoint packings of stars and cycles with a focus on their parameterized complexity:

Definition 1. Given a graph G and two integers k and s, Edge-Disjoint k-Packing of s-Stars (respectively, Edge-Disjoint k-Packing of s-Cycles) is the problem of deciding whether G contains at least k edge-disjoint copies of s-stars (respectively, s-cycles).

We first review some preliminaries on parameterized complexity [8,15]. A *parameterized problem* is a language of the form (x, k), where x is the input instance, and k is a nonnegative integer called the *parameter*. A parameterized problem is *fixed parameter tractable* (FPT) if it can be solved in $f(k) \cdot |x|^{O(1)}$ time, where f is a computable function solely dependent on k, and $|x|$ is the size of the input instance. On the other hand, many problems that are not known to be fixed-parameter tractable can be classified in an infinite hierarchy of complexity classes W[1] $\subseteq$ W[2] $\subseteq \cdots$. For example, with the solution

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size as parameter, VERTEX COVER is fixed-parameter tractable, CLIQUE and INDEPENDENT SET are W[1]-complete, and DOMINATING SET is W[2]-complete. An *FPT reduction* from a parameterized problem X to another parameterized problem Y is an algorithm that transforms any instance (x,k) of X to an instance (y,l) of Y with $l \le g(k)$ in $f(k) \cdot |x|^{O(1)}$ time, where f and g are computable functions solely dependent on k, such that (x,k) is a yes instance of X if and only if (y,k') is a yes instance of Y.

An instance (x,k) of a parameterized problem may be reduced to a *problem kernel*, that is, transformed by a polynomial-time algorithm to an equivalent reduced instance (x',k') with $\max\{|x'|,k'\} \le g(k)$ for some function g solely dependent on k. We call such a transformation the *kernelization* process, and call the function g the *size* of the kernel. When g is a polynomial function, we say that the problem has a *polynomial kernel*. It is well-known that a decidable parameterized problem is fixed-parameter tractable if and only if it is kernelizable.

Previous work. When $s \ge 3$, both EDGE-DISJOINT k-PACKING OF s-STARS and EDGE-DISJOINT k-PACKING OF s-CYCLES are NP-complete even in planar graphs [18,9,17]. On the other hand, when $s = 2$, EDGE-DISJOINT k-PACKING OF s-STARS is solvable in linear time [20]. Also, when $s \ge 3$, both EDGE-DISJOINT k-PACKING OF s-STARS and EDGE-DISJOINT k-PACKING OF s-CYCLES are solvable in polynomial time in trees [17].

In terms of kernelization complexity, Mathieson, Prieto and Shaw [21] gave a $4k$ problem kernel for EDGE-DISJOINT k-PACKING OF s-CYCLES when $s = 3$, and asked whether problem kernels can be obtained for the problem when $s \ge 4$. This $4k$ kernel was recently improved to $3.5k$ by Yang [26]. Bodlaender, Thomassé and Yeo [5] studied a similar problem on edge-disjoint packing of cycles, where the cycles may have different, arbitrary lengths; they gave an $O(k \log k)$ kernel for this problem.

The vertex-disjoint versions of the two packing problems have also been studied. For VERTEX-DISJOINT k-PACKING OF s-STARS, Prieto and Sloper [23] gave an $O(k^2)$ kernel when $s \ge 3$ and a $15k$ kernel when $s = 2$. The kernel size for $s \ge 3$ was subsequently improved to $O(k^{1+\epsilon})$ for any $\epsilon > 0$ by Fellows et al. [11], and to $O(k)$ by Xiao [25]. The kernel size for $s \ge 2$ was subsequently improved to $7k$ by Wang et al. [24], and to $6k$ by Chen et al. [6]. Prieto and Sloper [23] posed as an open problem whether problem kernels can be obtained for the edge-disjoint version of s-star packing with $s \ge 3$. For VERTEX-DISJOINT k-PACKING OF s-CYCLES, a result of Fellows et al. [12] implies an $O(k^3)$ kernel when $s = 3$, which was later improved to $45k^2$ by Moser [22]. Guo and Niedermeier [16] gave a $732k$ kernel in planar graphs when $s = 3$.

Our contributions. A general result on edge-disjoint graph packing [7] implies that EDGE-DISJOINT k-PACKING OF s-STARS and EDGE-DISJOINT k-PACKING OF s-CYCLES can be solved in $2^{O(ks)}$ time,[1] and hence are fixed-parameter tractable with both k and s as parameters. On the other hand, the NP-hardness of the two problems for any constant $s \ge 3$ [18,9,17] implies that unless P = NP, they are not FPT with only s as parameter.

We show that with only k as parameter, the two problems are both W[1]-hard and hence are not FPT either, unless FPT = W[1]. Moreover, EDGE-DISJOINT k-PACKING OF s-CYCLES is W[1]-hard even in bipartite graphs when parameterized by k. We also study the classical computational complexity of EDGE-DISJOINT k-PACKING OF s-CYCLES in specific graph classes, and show that it remains NP-complete when restricted to bipartite graphs and balanced 2-interval graphs.

The $2^{O(ks)}$-time algorithms for the two problems imply that they have kernels with size exponential in both k and s. It is natural to ask whether kernels with size polynomial in k and s are possible. Fellows et al. gave an $O(k^r)$ kernel for r-SET PACKING [13, Lemma 6], which implies an $O(k^s)$ kernel[2] for the general graph packing problem of determining whether a graph G contains k edge-disjoint copies of another graph H with s vertices, for any fixed s. The result of Fellows et al. [13] does not imply an $O(k^s)$ kernel for the general graph packing problem with both k and s as parameters, because the enumeration of all copies of H (s-stars or s-cycles in our setting) into s-sets requires $n^{O(s)}$ preprocessing time. Also, the size of the implied kernel is only polynomial in k when s is fixed.

We show that EDGE-DISJOINT k-PACKING OF s-STARS in fact has a kernel with size polynomial in both k and s, but in contrast, unless NP $\subseteq$ coNP/poly, EDGE-DISJOINT k-PACKING OF s-CYCLES does not have a kernel with size polynomial in both k and s, and moreover does not have a kernel with size polynomial in s for any fixed k. Our kernelization results include a ks^2 kernel for EDGE-DISJOINT k-PACKING OF s-STARS in general graphs, a $96k^2$ kernel for EDGE-DISJOINT k-PACKING OF 4-CYCLES in general graphs, and a $96k$ kernel for EDGE-DISJOINT k-PACKING OF 4-CYCLES in planar graphs.

Our results are grouped into the next two sections, with the various hardness results in Section 2 and the polynomial kernelization results in Section 3.

[1] For the general problem of deciding whether a graph G contains k edge-disjoint copies of a graph H, the running time of the general algorithm [7, Corollary 5.7] is $4^{rk+O(\log^3(rk))}n^{r+1}$, where n and r are the numbers of vertices in G and H, respectively. The n^{r+1} factor in the running time accounts for the time for solving the subproblem of determining whether G contains H as a subgraph, by brute force. The brute-force algorithm for this subproblem can be replaced by faster algorithms in our setting, with polynomial running time when H is an s-star, or $2^{O(s)}\text{poly}(n)$ time when H is an s-cycle [2].

[2] We remark that in our setting of EDGE-DISJOINT k-PACKING OF s-CYCLES, since any two different s-cycles share at most $s-2$ edges, the base case of $f(0,k) = 1$ in the analysis of [13, page 170] can be upgraded to $f(1,k) = 1$, which saves 1 in the exponent and yields an $O(k^{s-1})$ kernel.