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Component connectivity of hypercubes

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ABSTRACT

The r-component connectivity $c\kappa_r(G)$ of a non-complete graph G is the minimum number of vertices whose deletion results in a graph with at least r components. In this paper, we determine the component connectivity of the hypercube $c\kappa_{r+1}(Q_n) = -\frac{r^2}{2} + (2n - \frac{5}{2})r - n^2 + 2n + 1$ for $n + 1 \le r \le 2n - 5$, $n \ge 6$. This paper extends the results in Hsu et al. (2012) [3].

1. Introduction

Let G be a non-complete graph. A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. The r-component connectivity $c\kappa_r(G)$ of a graph G is the size of the smallest r-component cut of G. By the definition of the $c\kappa_r(G)$, it can be seen that $c\kappa_{r+1}(G) \ge c\kappa_r(G)$ for every positive integer r.

An interconnection network is usually modeled by a connected graph in which vertices represent processors and edges links between processors. The usual connectivity $\kappa(G)$ of a graph is the minimum number of vertices whose deletion results in a disconnected graph. The connectivity is one of the important parameters to evaluate the reliability and fault tolerance of a network. The r-component connectivity is an extension of the usual connectivity $c\kappa_2(G)$. The r-component connectivity and r-component edge connectivity were introduced in [1] and [6] independently. In [3], Hsu et al. determined the r-component connectivity of the hypercube Q_n for $r = 2, 3, \cdots, n + 1$. In this paper, we determine the r-component connectivity of the hypercube Q_n for $r = n + 2, n + 3, \cdots, 2n - 4$. This result extends the result in [3].

The n-dimensional hypercube Q_n is an undirected graph $Q_n = (V, E)$ with $|V| = 2^n$ and $|G| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is an edge between two vertices whenever there binary string representation differs in only one bit position. The 3-dimensional and 4-dimensional hypercubes are shown in Fig. 1 and Fig. 2, respectively.

Following Latifi in [4], we express Q_n as $D_0 \bigodot D_1$, where D_0 and D_1 are two $n - 1$ cubes of Q_n induced by the vertices with the ith coordinates 0 and 1 respectively. Clearly, each vertex in Q_n has degree n. An independent vertex set is that every two vertices in the set are nonadjacent. Let v be a vertex of a graph G, we use $N_G(v)$ to denote the vertices that are adjacent to v. As Q_n is bipartite, the neighbor set of a vertex v is independent. Let $A \subseteq V(G)$, we denote by $N_G(A)$ the vertex set $\bigcup_{v \in V(A)} N_G(v) \setminus V(A)$ and $C_G(A) = N_G(A) \bigcup A$. For the related studies on the conditional connectivity of the hypercubes, we refer to [2,5,7,8,11,12].

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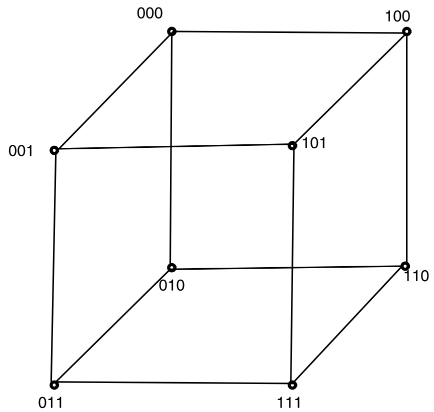

Fig. 1. The 3-dimensional hypercube.

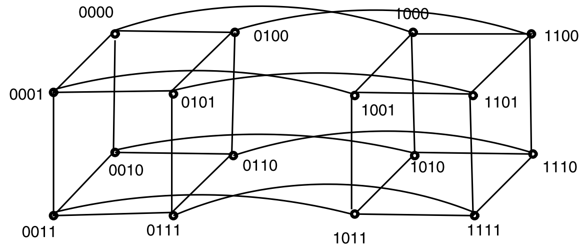

Fig. 2. The 4-dimensional hypercube.

2. Main results

Theorem 1. ([3]) *Let $n \geq 2$ and $1 \leq r \leq n$, then $c\kappa_{r+1} = rn - \frac{r(r+1)}{2} + 1$.*

Yang et al. in [10] introduced the following two quadratic functions which are defined as:

$$P_n(x) = -\frac{x^2}{2} + (n - \frac{1}{2})x + 1, 1 \leq x \leq n + 1,$$
$$Q_n(x) = -\frac{x^2}{2} + (2n - \frac{3}{2})x - (n^2 - 2), n + 2 \leq x \leq 2n,$$

where $P_n(x)$ and $Q_n(x)$ denote the minimum number of vertices adjacent to a set of x vertices in Q_n. In [7,10], the authors showed that if $X \subseteq N(u)$ with $|X| = x$ for some $u \in V(Q_n)$, then $|N_{Q_n}(X)| = P_n(x)$, where $1 \leq x \leq n$. Yang and Meng in [9] showed the following, which plays an important role in the proof.

Lemma 2. ([9]) *Assume that $n \geq 5$ and $A \subseteq Q_n$. If $|V(A)| \geq 2n$ and $|V(Q_n) - C_{Q_n(A)}| \geq |V(A)|$, then $|N_{Q_n}(A)| \geq Q_n(2n)$.*

Define $f_n(x) = -\frac{x^2}{2} + (2n - \frac{5}{2})x - n^2 + 2n + 1$, $n + 1 \leq x \leq 2n - 5$. Clearly, $f_n(x)$ is strictly monotonically increasing when $x \leq 2n - 3$.

Lemma 3. *$P_n(x) + P_n(a - x) \geq f_{n+1}(a)$ for $a - n \leq x \leq n$, $n + 1 \leq a \leq 2n$.*

Proof. Consider the quadratic function $g(x) = P_n(x) + P_n(a - x) - f_{n+1}(a) = -x^2 + ax - na + n^2$. As $a - n \leq x \leq n$, $g(x)$ achieves its minimum at $x = a - n$ or $x = n$. We derive $g(a - n) = -(a - n)^2 + a(a - n) - na + n^2 = 0$ and $g(n) = -n^2 + an - na + n^2 = 0$, so $g(x) \geq 0$ when $a - n \leq x \leq n$, $n + 1 \leq a \leq 2n$. □

Lemma 4. *$P_n(r_1) + f_n(r_2) \geq f_{n+1}(r)$ for $2 \leq r_1 \leq r - n - 1$, $r_2 \geq n + 1$ and $r_1 + r_2 = r \leq 2n - 3$.*

Proof. Note that $P_n(r_1) + f_n(r_2) - f_{n+1}(r) = -r_1^2 + (r - n + 2)r_1 + 2n - 2r$. Let $g(r_1) = -r_1^2 + (r - n + 2)r_1 + 2n - 2r$, $2 \leq r_1 \leq r - n - 1$. Then $g(r_1)$ is minimized at $r_1 = 2$ or $r_1 = r - k - 1$. As $g(2) = 0$, $g(k - n - 1) = 0$, so the result holds. □