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Strong matching preclusion for torus networks ☆

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ABSTRACT

The torus network is one of the most popular interconnection network topologies for massively parallel computing systems. Strong matching preclusion that additionally permits more destructive vertex faults in a graph is a more extensive form of the original matching preclusion that assumes only edge faults. In this paper, we establish the strong matching preclusion number and all minimum strong matching preclusion sets for bipartite torus networks and 2-dimensional nonbipartite torus networks.

1. Introduction

A matching of a graph is a set of pairwise nonadjacent edges. For a graph with n vertices, a matching M is called perfect if its size $|M| = \frac{n}{2}$ for even n, or almost perfect if $|M| = \frac{n-1}{2}$ for odd n. A graph is matchable if it has either a perfect matching or an almost perfect matching. Otherwise, it is called unmatchable. Throughout the paper, we only consider simple and even graphs, that is, graphs with an even number of vertices with no parallel edges or loops. For graph-theoretical terminology and notation not defined here we follow [4]. Let $G = (V(G), E(G))$ be a graph. A set F of edges in G is called a matching preclusion set (MP set for short) if $G - F$ has neither a perfect matching nor an almost perfect matching. The matching preclusion number of G (MP number for short), denoted by $mp(G)$, is defined to be the minimum size of all possible such sets of G. The minimum MP set of G is any MP set whose size is $mp(G)$. A matching preclusion set of a graph is trivial if all its edges are incident to a single vertex.

Since the problem of matching preclusion was first presented by Brigham et al. [3], several classes of graphs have been studied to understand their matching preclusion properties [5–8,11,13,14]. An obvious application of the matching preclusion problem was addressed in [3]: when each node of interconnection networks is demanded to have a special partner at any time, those that have larger matching preclusion numbers will be more robust in the event of link failures.

Another form of matching obstruction, which is in fact more offensive, is through node failures. As an extensive form of matching preclusion, the problem of strong matching preclusion was proposed by Park and Ihm in [12]. A set F of vertices and/or edges in a matchable graph G is called a strong matching preclusion set (SMP set for short) if $G - F$ has neither a perfect matching nor an almost perfect matching. The strong matching preclusion number (SMP number for short) of G, denoted by $smp(G)$, is defined to be the minimum size of all possible such sets of G. The minimum SMP set of G is any

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SMP set whose size is $smp(G)$. Note that the strong matching preclusion is more general than the problems discussed in [1,9], which considered only vertex deletions.

Specially, when G itself does not contain perfect matchings or almost perfect matchings, both $smp(G)$ and $mp(G)$ are regarded as zero. These numbers are undefined for a trivial graph with only one vertex. Notice that an MP set of a graph is a special SMP set of the graph.

Proposition 1.1. *(See [12].) For every nontrivial graph G, $smp(G) \leqslant mp(G)$.*

However, the strong matching preclusion numbers did not decrease for such graphs as restricted hypercube-like graphs and recursive circulants [12]. Then, followed by this work, the strong matching preclusion problem was studied for some classes of graphs such as alternating group graphs and split-stars [2].

When a set F of vertices and/or edges is removed from a graph, the set is called a fault set. Let F_v and F_e be the fault vertex set and the fault edge set, respectively. We have $F = F_v \cup F_e$. For any vertex $v \in V(G)$, let $N_G(v)$ be all neighbouring vertices adjacent to v and let $I_G(v)$ be all edges incident to v. Clearly, a fault set, which separates exactly one isolated vertex from the remaining even graph, forms a simple SMP set of the original graph.

Proposition 1.2. *(See [12].) Let G be a graph. Given a fault vertex set $X(v) \subseteq N_G(v)$ and a fault edge set $Y(v) \subseteq I_G(v)$, $X(v) \cup Y(v)$ is an SMP set of G if* (i) *$w \in X(v)$ if and only if $(v, w) \notin Y(v)$ for every $w \in N_G(v)$, and* (ii) *the number of vertices in $G - (X(v) \cup Y(v))$ is even.*

The above proposition suggests an easy way of building SMP sets. Any SMP set constructed as specified in Proposition 1.2 is called trivial. If $smp(G) = \delta(G)$, then G is called maximally strong matched. If every minimum SMP set of G is trivial, then G is called super strong matched. It is easy to see that, for an arbitrary vertex of degree at least one, there always exists a trivial SMP set which isolates the vertex. This observation leads to the following fact.

Proposition 1.3. *(See [12].) For any graph G with no isolated vertices, $smp(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

2. Definitions and terminology

The torus forms a basic class of interconnection networks. Let G and H be two simple graphs. Their Cartesian product $G \times H$ is the graph with vertex set $V(G) \times V(H) = \{gh\colon\ g \in V(G),\ h \in V(H)\}$, in which two vertices g_1h_1 and g_2h_2 are adjacent if and only if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $(g_1, g_2) \in E(G)$ and $h_1 = h_2$. For $n \geqslant 3$, let $G_1, G_2, \ldots, G_n$ be n simple graphs. Similarly, the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ can be defined. It is easy to see that "×" is associative and commutative under isomorphism. Let C_k be the cycle of length k with the vertex set $\{0, 1, \ldots, k-1\}$. Two vertices $u, v \in V(C_k)$ are adjacent in C_k if and only if $u = v \pm 1 \pmod{k}$. The torus $T(k_1, k_2, \ldots, k_n)$ with $n \geqslant 2$ and $k_i \geqslant 3$ for all i is defined to be $T(k_1, k_2, \ldots, k_n) = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}$ with the vertex set $\{u_1u_2 \ldots u_n\colon\ u_i \in \{0, 1, \ldots, k_i - 1\},\ 1 \leqslant i \leqslant n\}$. Two vertices $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ are adjacent in $T(k_1, k_2, \ldots, k_n)$ if and only if there exists some $j \in \{1, 2, \ldots, n\}$ such that $u_j = v_j \pm 1 \pmod{k_j}$ and $u_i = v_i$ for $i \in \{1, 2, \ldots, n\} \backslash \{j\}$. Clearly, $T(k_1, k_2, \ldots, k_n)$ is a connected $2n$-regular graph consisting of $k_1k_2 \ldots k_n$ vertices. Note that we only consider even graphs in this paper, which implies that at least one of $k_1, k_2, \ldots, k_n$ is even.

Let $T(k_1, k_2)$ be a 2-dimensional torus, where $k_1 \geqslant 3$ and $k_2 \geqslant 3$. Then $T(k_1, k_2) = C_{k_1} \times C_{k_2}$. We view $C_{k_1} \times C_{k_2}$ as consisting of k_2 copies of C_{k_1}. Let these copies be $C_{k_1}^0, C_{k_1}^1, \ldots, C_{k_1}^{k_2-1}$ labeled along the cycle C_{k_2}. The edges between different copies of C_{k_1} are called cross edges. Denote the set of cross edges between $C_{k_1}^i$ and $C_{k_1}^{i+1 (\mathrm{mod}\ k_2)}$ by $M_{i,i+1 (\mathrm{mod}\ k_2)}$ for $0 \leqslant i \leqslant k_2 - 1$. For clarity of presentation, we omit writing "(mod k_2)" in similar expressions for the remainder of the paper. Clearly, each of these sets is a matching saturating all vertices of the corresponding copies of C_{k_1}. For convenience, a vertex with subscript 0 (e.g. x_0) will denote a vertex in $C_{k_1}^0$, the corresponding vertex with subscript 1 (e.g. x_1) will denote the vertex in $C_{k_1}^1$ which is adjacent to this vertex via a cross edge, etc., and the corresponding vertex with subscript $k_2 - 1$ (e.g. x_{k_2-1}) will denote the vertex in $C_{k_1}^{k_2-1}$ which is adjacent to this vertex via a cross edge. The vertices $x_0, x_1, \ldots, x_{k_2-1}$ and the cross edges between them form a cycle of length k_2, which is denoted by $C_{k_2}(x_i)$ for some $i \in \{0, 1, \ldots, k_2 - 1\}$. For any matching M_i in $C_{k_1}^i$, the matching M_j, which satisfies that $(x_j, y_j) \in M_j$ if and only if $(x_i, y_i) \in M_i$, is called the corresponding matching to M_i.

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. The length of a path is the number of its edges. The path is odd or even according to the parity of its length. For notational simplicity, denote by $|G|$ the number of vertices in a graph G. Let G_1 and G_2 be two graphs. $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.