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Solving the initial value problem of discrete geodesics☆

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HIGHLIGHTS

- Shortest geodesic is not able to solve the initial value problem of discrete geodesic.
- Geodesic equation are second-order ODEs.
- We solve the initial value problem on triangle meshes by solving a first-order ODE
- The computed discrete geodesic path converges to the one on the smooth surface.

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ABSTRACT

Computing geodesic paths and distances is a common operation in computer graphics and computer-aided geometric design. The existing discrete geodesic algorithms are mainly designed to solve the boundary value problem, i.e., to find the shortest path between two given points. In this paper, we focus on the initial value problem, i.e., finding a uniquely determined geodesic path from a given point in any direction. Since the *shortest* paths do not provide the unique solution on triangle meshes, we solve the initial value problem in an indirect manner: given a fixed point and an initial tangent direction on a triangle mesh M, we first compute a geodesic curve $\widehat{\gamma}$ on a piecewise smooth surface $\widehat{M}$, which well approximates the input mesh M and can be constructed at little cost. Then, we solve a first-order ODE of the tangent vector using the fourth-order Runge–Kutta method, and parallel transport it along $\widehat{\gamma}$. When the geodesic curve reaches the boundary of the current patch, its tangent can be directly transported to the neighboring patch, thanks to the G^1-continuity along the common boundary of two adjacent patches. Finally, once the geodesic curve $\widehat{\gamma}$ is available, we project it onto the underlying mesh M, producing the discrete geodesic path γ, which is guaranteed to be unique on M. It is worth noting that our method is different from the conventional methods of directly solving the geodesic equation (i.e., a second-order ODE of the position) on piecewise smooth surfaces, which are difficult to implement due to the complicated representation of the geodesic equation involving Christoffel symbols. The proposed method, based on the first-order ODE of the tangent vector, is intuitive and easy for implementation. Our method is particularly useful for computing geodesic paths on low-resolution meshes which may have large and/or skinny triangles, since the conventional *straightest* geodesic paths are usually far from the ground truth.

1. Introduction

Computing geodesic distances and geodesic paths plays an important role in many fields, such as CAD/CAM [1], path planning [2], shape analysis [3], parameterization [4,5], segmentation [6], and medial axis [7]. Geodesics on smooth surfaces are well understood in classic differential geometry. However, the discrete geodesic problem, i.e., computing geodesic distances and paths on discrete domains such as polygonal meshes, is fundamentally different from its smooth counterpart, due to the difference between smooth and discrete domains. For example, geodesics is both straightest and locally shortest on smooth surfaces, but such a nice property does not hold on polygonal meshes. The discrete *shortest* geodesic is not equivalent to the discrete *straightest* geodesic, which bisects the vertex angles, since the former is a metric but the latter is not.

As a fundamental problem in computational geometry and geometric modeling, the discrete geodesic problem has been studied extensively in the past three decades. To date, many elegant algorithms have been proposed. Representative works include the

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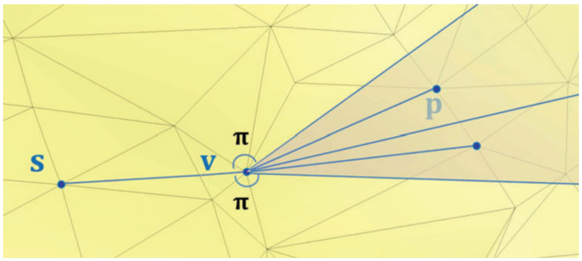

Fig. 1. The discrete *shortest* geodesic does not solve the initial value problem. Let v be a saddle vertex, whose curve angle is more than 2π. When a shortest geodesic path, say $\gamma(s, v)$, passes through v, it splits into many outgoing geodesic paths: any line segment $\overline{pv}$ in the fan-shaped area (in gray) together with $\gamma(s, v)$, is a shortest path from s to p. Therefore, the initial value problem does not have a unique solution, if one considers the *shortest* geodesic paths.

exact[1] algorithms (e.g., the Mitchell–Mount–Papadimitriou (MMP) algorithm [8] and the Chen–Han (CH) algorithm [9]), the PDE methods (e.g., the fast marching method [10] and the heat method [11,12]), and the graph-theoretic methods (e.g., the saddle vertex graph method [13]). These algorithms, however, are mainly designed to solve the boundary value problem, that is, to find the shortest path between two fixed endpoints.

Mitchell et al. [8] proved that the general form of a *shortest* geodesic path γ was an alternating sequence of vertices and (possibly empty) edges. The unfolded image of the path along any edge sequence is a straight line segment, and the curve angle of any vertex which γ passes through is greater than or equal to π. As Fig. 1 shows, when a shortest geodesic path γ passes through a saddle vertex (whose curve angle is more than 2π), γ splits into multiple outgoing geodesic paths. Therefore, the shortest geodesic paths, although well defined, are not able to solve the initial value problem of discrete geodesics, which aims at finding the *unique* geodesic path from a fixed point and in a given tangent direction.

In this paper, we present a method for solving the initial value problem on triangle meshes. To ensure a unique solution, we adopt an indirect strategy. Given a fixed point and an initial tangent direction on a triangle mesh M, we first compute a geodesic curve $\widehat{\gamma}$ on a piecewise smooth surface $\widehat{M}$, which well approximates the input mesh M and can be constructed at little cost. Then, we solve a first-order ODE of the tangent vector by the fourth-order Runge–Kutta method, and parallel transport it along $\widehat{\gamma}$. When the geodesic curve reaches the boundary of the current patch, its tangent vector can be directly transported to the neighboring patch, thanks to the G^1-continuity along the common boundary of two adjacent patches. Finally, once the geodesic curve $\widehat{\gamma}$ is available, we project it onto the underlying mesh M, producing the discrete geodesic path γ, which is guaranteed to be unique on the triangle mesh M. See Fig. 2.

It is worth noting that our method is different from the conventional methods of directly solving the geodesic equation (i.e., a second-order ODE of the position) on the piecewise smooth surface, which are tedious and difficult to implement, due to the complicated representation of the geodesic equation involving Christoffel symbols. Based on the first-order ODE of the tangent vector, the proposed method is intuitive and easy to implement. We observe that our method is particularly useful for computing geodesic paths on low-resolution meshes with large and/or skinny triangles, where the conventional *straightest* geodesic paths are usually far from the ground truth. In addition, our method can be easily adapted to work on non-orientable surfaces.

[1] If the numerical computation is exact, the computed geodesic distance is also exact.

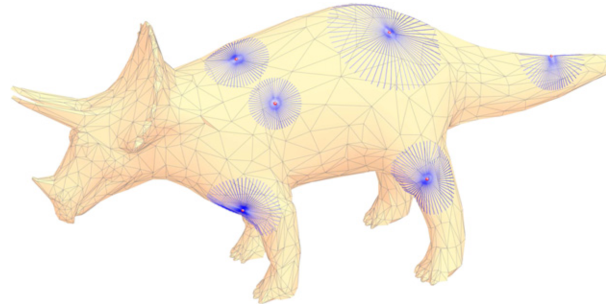

Fig. 2. Our method solves the initial value problem of discrete geodesics. For each point (in red) on the Rhino model, we compute geodesic paths in 60 tangent directions, which are evenly sampled on the tangent plane. Each tangent direction is guaranteed to produce a unique geodesic path on the triangle mesh. Our method is numerically stable and works well on meshes with large and/or skinny triangles. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2. Related work

This section presents the related work on computing geodesic paths on discrete domains and discrete differential geometry. As mentioned above, discrete geodesic paths are fundamentally different from geodesics on smooth surfaces, since the shortest geodesics and the straightest geodesics are not equivalent to each other any longer.

Shortest geodesics have been extensively studied and also widely used in computer graphics community. Existing methods for computing exact shortest geodesic paths on polyhedral surfaces can be generally grouped into two categories, namely, the MMP algorithm and the CH algorithm. Both methods are developed based on the continuous Dijkstra's algorithm, that iteratively propagates the discrete wavefront from the source to the destination. They differ in the wavefront organization and propagation scheme. The MMP algorithm has an $O(n^2 \log n)$ time complexity and an $O(n^2)$ space complexity for a mesh with n vertices. The CH algorithm runs in $O(n^2)$ time and takes $O(n)$ space. Different extensions on the two algorithms have been developed, which aim at parallelization [14], performance improvement [15–18] and robustness [19], computing geodesic offsets [20,21], geodesic loops [22], and all-pairs geodesics [23].

Straightest geodesics receives relatively less attention than shortest geodesics. Polthier and Schmies [24] introduced the discrete geodesic curvature and defined the straightest geodesic on polyhedral surface as a path that has equal curve angle on both sides at each point. Then, they proposed the geodesic Euler method and the geodesic Runge–Kutta method for integrating a given vector field on a surface. Polthier and Schmies also developed the geodesic flow method [25] to compute the evolution of the front of a point wave on a polyhedral surface. At each time step, the front is a topological circle on the surface [26], where each point moves a constant distance in orthogonal direction to the curve by the straightest geodesic path. Kumar et al. [27] observed that the straightest geodesic obtained by tracing the path with equal left and right curve angles was far from the correct geodesic curve on the smooth surface. Therefore, they proposed a sectional plane method, which takes into account the variation of the tessellation normal along the geodesic path. Kumar et al.'s method can be considered as an extrinsic Euler method, which solves the geodesic equation with the first-order approximation. Therefore, their method tends to suffer from serious numerical issue and may deviate from the true geodesic curve after only a few iterations. Based on the fast marching method and the straightest geodesic [24], Martinez et al. [28] proposed an iterative algorithm for computing the shortest path between two fixed points.