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Re-parameterization reduces irreducible geometric constraint systems ☆

Hichem Barki [a,*], Lincong Fang [b], Dominique Michelucci [c], Sebti Foufou [a]

[a] CSE Department, College of Engineering, Qatar University, PO BOX 2713, Doha, Qatar

[b] School of Information Technology, Zhejiang University of Finance & Economics, 310018 Hangzhou, China

[c] LE2I UMR6306, CNRS, Arts et Métiers, Université Bourgogne Franche-Comté, F-21000 Dijon, France

HIGHLIGHTS

- A new re-parameterization for reducing and unlocking irreducible geometric systems.
- No need for the values of the key unknowns and no limit on their number.
- Enabling the usage of decomposition methods on irreducible re-parameterized systems.
- Usage at the lowest linear Algebra level and significant performance improvement.
- Benefits for numerous solvers (Newton–Raphson, homotopy, *p*-adic methods, etc.)

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ABSTRACT

You recklessly told your boss that solving a non-linear system of size n (n unknowns and n equations) requires a time proportional to n, as you were not very attentive during algorithmic complexity lectures. So now, you have only one night to solve a problem of big size (e.g., 1000 equations/unknowns), otherwise you will be fired in the next morning. The system is well-constrained and structurally irreducible: it does not contain any strictly smaller well-constrained subsystems. Its size is big, so the Newton–Raphson method is too slow and impractical. The most frustrating thing is that if you knew the values of a small number $k \ll n$ of key unknowns, then the system would be reducible to small square subsystems and easily solved. You wonder if it would be possible to exploit this reducibility, even without knowing the values of these few key unknowns. This article shows that it is indeed possible. This is done at the lowest level, at the linear algebra routines level, so that numerous solvers (Newton–Raphson, homotopy, and also p-adic methods relying on Hensel lifting) widely involved in geometric constraint solving and CAD applications can benefit from this decomposition with minor modifications. For instance, with $k \ll n$ key unknowns, the cost of a Newton iteration becomes $O(kn^2)$ instead of $O(n^3)$. Several experiments showing a significant performance gain of our re-parameterization technique are reported in this paper to consolidate our theoretical findings and to motivate its practical usage for bigger systems.

1. Introduction

Geometric modeling by constraints [1–7] leads to large systems of non-linear (algebraic most of the time) equations. In their seminal work, Gao et al. [8] automatically generated all the possible irreducible and structurally well-constrained 3D systems of geometric constraints (which they called basic configurations) that involve up to six geometric primitives (points, lines, and planes). These basic configurations correspond to 3D sub-problems that often occur in geometric constraint solving problems, in variational modeling, or in CAD/CAM applications. Most of the time, and contrarily to the 2D case, there is no closed-form solution for such 3D basic configurations. Gao et al. proposed the Locus Intersection Method (LIM) for solving these basic configurations and showed that among these possible 683 systems, 614 ones can be solved by using one key unknown (also called the driving parameter), while solving the remaining 69 ones requires two key

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* Corresponding author.
E-mail addresses: hbarki@qu.edu.qa (H. Barki), lincongfang@gmail.com (L. Fang), dmichel@u-bourgogne.fr (D. Michelucci), sfoufou@qu.edu.qa (S. Foufou).

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unknowns. They referred to these two re-parameterization solving methods as LIM1 and LIM2.

Re-parameterization consists in identifying or introducing a small number of *key unknowns*, also called *parameters* in the literature (hence the re-parameterization term), which have the following property: "*if the values of these key unknowns were known, then the system would be reducible to much smaller structurally irreducible subsystems and thus it would be easily solved*".

The work presented in [9] relied on properties of bipartite graphs underlying systems of equations, to polynomially decompose large systems into well-, over-, and under-constrained subsystems. The same paper [9] also proposed an efficient method to decompose or reduce well-constrained systems into irreducible and well-constrained (having as many equations as unknowns) subsystems. These decompositions have considerably speeded-up the solving process, and also allowed debugging systems of constraints in a constraint programming context. However, the reduction methods proposed in [9] have limitations. For instance, they do not apply to re-parameterized systems proposed in [8,10–12]. This inability to reduce re-parameterized systems is due to the fact that the methods of [9] are unable to reduce irreducible systems, and that re-parameterized systems are irreducible. Later on, after the locus intersection method of Gao et al. [8] became popular, several techniques for the decomposition of geometric systems with re-parameterization have been proposed [12,10,11]. These methods decompose well-constrained 3D systems into re-parameterized subsystems with a small set of key unknowns per subsystem, perform in polynomial time, and provide sub-optimal but good results.

In spite of the breakthrough made by the re-parameterization technique and the aforementioned methods seeking to find small sets of key unknowns, there are two major limitations. First of all, since re-parameterized systems are irreducible, the decomposition methods proposed in [9] cannot apply to them. Second, with basic configurations involving more than six geometric primitives or for systems of geometric constraints involving more complex geometric primitives (cylinders, spheres, cones, tori, etc.), using one or two key unknowns is not enough, even when employing the best re-parameterization techniques known so far [12]. In other words, although LIM1 (Locus Intersection method with one key unknown) is very fast and simple, its variants LIM*k* (Locus Intersection methods involving $k \geq 2$ key unknowns) become much less convenient.

Contribution. Our work addresses the aforementioned major limitations of re-parameterization/decomposition techniques. It proposes a technique for efficiently reducing or unlocking irreducible re-parameterized systems of equations like those proposed in [8,10–12] and resulting from geometric constraint systems and geometric modeling applications, so that the decomposition methods proposed in [9], which were unable to reduce such re-parameterized irreducible systems, become applicable. Furthermore, this work shows that it is possible to benefit from these decomposition techniques even when the values of the key unknowns are not known and the number of these key unknowns k is greater than 2. For this purpose, we propose to exploit re-parameterization at the lowest level, which is the level of the underlying linear algebra routines: solving a linear system or inverting a matrix. The focus on the lowest level for exploiting re-parameterization is not hazardous. It is pertinent and highly motivated by the fact that most existing solvers, like Newton–Raphson or homotopy rely on the aforementioned low-level linear algebra routines. Consequently, this level seems to be the best place to exploit re-parameterization. Although, doing so does not prevent using re-parameterization at some higher level.

Our paper focuses on exploiting the re-parameterization technique for reducing and thus efficiently solving well-constrained irreducible re-parameterized systems which are determined in advance and for which the key unknowns or parameters (even if their values are unknown) are already identified. It does not seek to find the best decomposition or re-parameterization of a system, a problem that has already been investigated in the literature [12] (cf. Section 10).

Although this paper focuses on solving well-constrained irreducible re-parameterized systems of equations and not on directly solving (under-constrained) geometric constraint systems involved in geometric modeling, the latter easily translate into well-constrained systems of equations which are perfectly handled by our technique. For instance, all the basic configurations enumerated in [8] can be solved by our technique which goes beyond the locus intersection method as the latter is limited to one or two key unknowns, while we do not have such limitation. Other examples of geometric constraint systems can be found in [12], while the particular case of the pentahedron problem and the way it is more efficiently solved through re-parameterization are discussed in Section 8.

The rest of this paper is organized as follows: we first introduce the re-parameterization technique in Section 2 through examples, with a particular emphasis on the LIM involving one parameter. In Section 3, we briefly present matching theory and show how combinatorial decomposition methods are applied in order to improve the performance of linear algebra routines. After that, we show in Section 4 how decomposition speeds-up linear algebra routines. In Section 5, we show how re-parameterization speeds-up linear algebra routines for re-parameterized systems, so that Newton and homotopy methods can straightforwardly benefit from re-parameterization. This section also draws a complexity study. Section 6 explains how Hensel lifting in *p*-adic methods can take advantage of re-parameterization as well. This is an important result as it shows that not only numerical analysis, but symbolic computations, like Gröbner bases, may also benefit from re-parameterization. Section 7 shows that interval solvers may also benefit from re-parameterization, however the wrapping effect requires further research. Section 8 presents an experimental study of the performance of our re-parameterization technique at the lowest level of linear algebra routines involved in numerous solvers (Newton–Raphson, homotopy, and also *p*-adic methods relying on Hensel lifting) and shows an important speed-up. This section also presents a CAD example showing the benefits of our re-parameterization technique when applied to geometric constraint systems. Section 9 examines the issues of using re-parameterization at a higher level. Finally, Section 10 presents future works and open questions before Section 11 concludes the paper.

2. Understanding re-parameterization

In this section, we first illustrate the re-parameterization technique by means of two examples in 2D and 3D.

2.1. A trivial 2D example

Fig. 1 depicts a system of geometric constraints in 2D. For this system, the lengths of all the edges are given. It is easy to see that this system is under-constrained because it involves twelve unknowns (2D coordinates of its six vertices) and nine (distance) constraints. To make this system well-constrained, we employ placement rules commonly used in the literature, to constrain the placement of a particular subset of a geometric system [13], and transform it into a well-constrained system, without affecting the set of possible solutions. For our 2D system, we fix three coordinates in 2D, which is equivalent to fixing the positions of the three points of one triangle, say $A'B'C'$. Point $A'(x_{A'} = 0, y_{A'} = 0)$ is placed at the coordinates origin, point $B'(x_{B'} > 0, y_{B'} = 0)$ is placed on the positive x-axis, and point $C'(x_{C'}, y_{C'} > 0)$ is placed in