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Comparison of discrete Hodge star operators for surfaces

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A R T I C L E I N F O

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A B S T R A C T

We investigate the performance of various discrete Hodge star operators for discrete exterior calculus (DEC) using circumcentric and barycentric dual meshes. The performance is evaluated through the DEC solution of Darcy and incompressible Navier–Stokes flows over surfaces. While the circumcentric Hodge operators may be favorable due to their diagonal structure, the barycentric (geometric) and the Galerkin Hodge operators have the advantage of admitting arbitrary simplicial meshes. Numerical experiments reveal that the barycentric and the Galerkin Hodge operators retain the numerical convergence order attained through the circumcentric (diagonal) Hodge operators. Furthermore, when the barycentric or the Galerkin Hodge operators are employed, a super-convergence behavior is observed for the incompressible flow solution over unstructured simplicial surface meshes generated by successive subdivision of coarser meshes. Insofar as the computational cost is concerned, the Darcy flow solutions exhibit a moderate increase in the solution time when using the barycentric or the Galerkin Hodge operators due to a modest decrease in the linear system sparsity. On the other hand, for the incompressible flow simulations, both the solution time and the linear system sparsity do not change for either the circumcentric or the barycentric and the Galerkin Hodge operators.

1. Introduction

Discrete exterior calculus (DEC) is a paradigm for the numerical solution of partial differential equations (PDEs) on simplicial meshes [1,2]. A main advantage of DEC is the mimetic behavior of its discrete operators where they retain at the discrete level many of the identities/rules of their smooth counterparts. Over the past decade, DEC was used to numerically solve many physical problems including Darcy [3,4] and incompressible flows [5–7]. The mimetic behavior of the DEC discrete operators generally results in superior conservation properties for DEC discretizations. There also exist other numerical methods to discretize vector PDEs on surfaces that are not based on DEC [8–12].

The definition of most DEC operators requires a dual mesh related to the primal simplicial mesh. A common choice for the dual mesh is the circumcentric dual. The mutual orthogonality of the primal simplices and their circumcentric duals results in simple expressions for the discrete Hodge star operators. However, using the circumcentric dual limits DEC to only Delaunay simplicial meshes. In the case of non-Delaunay meshes, DEC implementations using the circumcentric dual along with the diagonal Hodge star definition yield incorrect numerical results [13]. Enabling DEC to handle arbitrary simplicial meshes is advantageous not only due to the extra flexibility in mesh generation but also due to significantly facilitating any subsequent local/adaptive mesh subdivision. This is important considering that successive subdivision of a Delaunay triangulation with obtuse-angled triangles would result in a non-Delaunay mesh.

An alternative choice for the dual mesh is the barycentric dual. Since the barycenter of a simplex always lies in its interior (unlike circumcenters), the dual barycentric cells are always non-overlapping for arbitrary simplicial meshes. However, the orthogonality between the primal and dual mesh objects that characterized the circumcentric dual is no longer valid for the barycentric dual. This implies that the DEC operators involving metrics may need to be redefined. For DEC applications over surface simplicial meshes, it becomes essential to redefine the Hodge star operator $*_1$ and its inverse $*_1^{-1}$. Two previous discrete definitions for Hodge operators on general simplicial meshes are the Galerkin [14] and the

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geometrical barycentric-based [15,16] definitions. By barycentric-based definitions here we mean the Hodge star definitions that are applicable when a barycentric dual mesh is employed. Both the two aforementioned definitions result in a sparse (but non-diagonal) matrix representation for the Hodge star operator $*_1$. Such a non-diagonal matrix structure further complicates the representation of the inverse operator $*_1^{-1}$. However, it is worth pointing out that on surface simplicial meshes, the DEC discretization of common partial differential equations; e.g. Poisson equation and incompressible Navier–Stokes equations, may not require the inverse operator $*_1^{-1}$. An alternative approach for dealing with non-Delaunay meshes, for the specific case of a scalar Laplacian, is through the intrinsic Delaunay triangulation [17].

The purpose of this study is to evaluate the performance of the barycentric-based and the Galerkin Hodge operators compared with the circumcentric (diagonal) Hodge star operator. The comparison is carried out through the DEC discretization of Darcy and incompressible Navier–Stokes flows over surfaces. The main differences between the circumcentric versus the barycentric dual meshes are first addressed in Section 2. The definitions of various Hodge star operators are then provided in Section 3. This is followed by numerical experiments for Darcy flow and incompressible Navier–Stokes equations in Section 4, demonstrating the behavior of the considered Hodge star operators. The results demonstrate the numerical convergence as well as the linear system sparsity and the computational cost for the various operators definitions considered during the current study. The paper closes with conclusions emphasizing the main observations and discussing related future insights.

2. Circumcentric versus barycentric dual meshes

Consider a domain Ω approximated with the simplicial complex K. This paper focuses only on simplicial meshes over flat/curved surfaces, and therefore the domain is considered to have a dimension $N = 2$. A k-simplex $\sigma^k \in K$ is defined by the nodes v_i forming it as $\sigma^k = [v_0, \dots, v_k]$. The definition of many DEC discrete operators requires the dual complex $\star K$ defined over the primal simplicial complex K. The dual to a primal simplex σ^k is the $(N-k)$-cell denoted by $\star\sigma^k \in \star K$. The top dimensional k-simplices/cells are consistently oriented; e.g. counterclockwise, in all our examples here.

Defined on the primal and dual mesh complexes are the spaces of the discrete k-forms denoted by $C^k(K)$ and $D^k(\star K)$, respectively. For $N = 2$, the spaces of the discrete forms are related via the discrete exterior derivative d_k and the discrete Hodge star $*_k$ operators as shown in the following diagram

$$\begin{array}{ccccc} C^0(K) & \xrightarrow{\mathrm{d}_0} & C^1(K) & \xrightarrow{\mathrm{d}_1} & C^2(K) \\ {\scriptstyle *_0^{-1}}\uparrow\downarrow{\scriptstyle *_0} & & {\scriptstyle *_1^{-1}}\uparrow\downarrow{\scriptstyle *_1} & & {\scriptstyle *_2^{-1}}\uparrow\downarrow{\scriptstyle *_2} \\ D^2(\star K) & \xleftarrow{-\mathrm{d}_0^T} & D^1(\star K) & \xleftarrow{\mathrm{d}_1^T} & D^0(\star K) \end{array} \quad (1)$$

where the superscript T indicates the matrix transpose.

A common choice for the dual complex is the circumcentric dual. This choice is motivated by the orthogonality between the primal and dual mesh objects, which simplifies the discrete Hodge star operators definitions. For example, for a smooth 1-form u, the orthogonality of the primal edge σ^1 and its dual $\star\sigma^1$ makes the component of u evaluated along σ^1 equal to the component of $*u$ evaluated along $\star\sigma^1$. This results globally in a diagonal matrix representation for the discrete Hodge star operator $*_1$ and also its inverse $*_1^{-1}$, simplifying various DEC computations.

Although the circumcentric duality yields this simplicity in the discrete Hodge star representation, this dual works correctly only on Delaunay meshes. Fig. 1(a) shows a sample non-Delaunay mesh. While the dual edges are, by the current convention, oriented 90° counterclockwise with respect to their primal edges, the dual to the primal edge $[v_1, v_2]$ is oriented in the opposite direction. This is due to the circumcenters of the neighboring triangles being in the reversed order. According to the notation in [13], this implies that the dual edge $\star[v_1, v_2]$ has a negative volume. An additional concern is the construction of the dual areas. For a non-Delaunay mesh, the circumcentric dual areas overlap with some of the areas sectors having negatively-signed volumes. For example, for the mesh in Fig. 1(a), the areas dual to the primal nodes v_0 and v_3 are overlapping; i.e. $\star v_0 \cap \star v_3 \neq \emptyset$. In addition, for the area dual to the node v_1, the part of $\star v_1$ that does not overlap with $\star v_0$ and $\star v_3$ has a positive volume, while the part that overlaps with $\star v_0$ and $\star v_3$ has a negative volume, according to the volume calculation convention defined in [13]. Previous analysis showed that the DEC numerical solution of Poisson equation over a non-Delaunay mesh, using the diagonal definition of the Hodge star operator, leads to incorrect results [13].

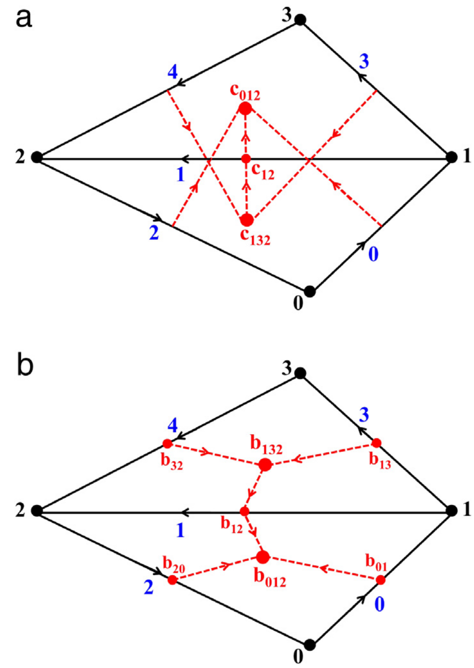

Fig. 1. Sketch for a sample non-Delaunay mesh with: (a) the circumcentric dual, and (b) the barycentric dual. The primal mesh is in black color and the dual mesh is in red color. $\mathbf{c}_{ijk}$ and $\mathbf{b}_{ijk}$ are the circumcenter and barycenter of the triangle $[v_i, v_j, v_k]$, respectively. $\mathbf{c}_{ij}$ and $\mathbf{b}_{ij}$ are the circumcenter and barycenter of the edge $[v_i, v_j]$, respectively.

An alternative choice for the dual mesh is the barycentric dual which is well defined on arbitrary simplicial meshes, as illustrated in Fig. 1(b). The dual to a primal triangle is its barycenter, the dual to a primal edge is the kinked line connecting the barycenters of the two neighboring triangles through the barycenter of the primal edge itself, and the dual to a primal node is the polygonal area formed by the duals of the primal edges connected to the primal node. Since the barycenter of each triangle is always in its interior, the dual barycentric cells do not overlap for arbitrary simplicial meshes. On the other hand, it is evident from Fig. 1 that the main difference between the circumcentric and the barycentric duals is the lack of mutual orthogonality between the primal and dual edges in the case of the barycentric dual.

The lack of orthogonality between the primal edges and their barycentric duals invalidates the diagonal representation of some Hodge star operators. This can be illustrated through numerical DEC experiments using the diagonal Hodge star operator constructed using a barycentric dual mesh. Fig. 2 shows the L^2 norm error for the Darcy flow and the incompressible