Computer Aided Geometric Design 45 (2016) 2–13

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Multivariate splines and the Bernstein–Bézier form of a polynomial

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ARTICLE INFO

Article history:
Available online 7 December 2015

ABSTRACT

This expository paper exhibits the power and versatility of the Bernstein–Bézier form of a polynomial, and the role that it has played in the analysis of multivariate spline spaces. Several particular applications are discussed in some detail. The purpose of the paper is to provide the reader with a working facility with the Bernstein–Bézier form.

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1. Introduction

The Bernstein–Bézier form of a polynomial, henceforth simply referred to as the *B-form*, has been essential and central in multivariate spline research during the past 40 years or so. The purpose of this article is to serve as a tutorial and illustrate the power and utility of the B-form. Several B-form based arguments will be shown in detail so that a reader new to the B-form should obtain a working familiarity with some of the techniques that proved useful in spline research. However, I'll make no attempt to give a comprehensive, or even just balanced, summary of the history or the state of the art. I'll quote only those references that have a direct bearing on the material discussed here. For a complete description of the subject and its history, a comprehensive list of references, and an account of the state of the art as of 2007, consult the authoritative monograph by Lai and Schumaker (2007).

Splines are smooth piecewise polynomial functions defined on a partition of an underlying domain $\Omega \subset \mathbb{R}^k$. Splines are called *univariate* if $k = 1$, *bivariate* if $k = 2$, *trivariate* if $k = 3$, and, in general, *multivariate* if $k > 1$. Throughout this paper, r will denote the degree of smoothness, d the polynomial degree, k the dimension of the underlying domain, and N the number of regions in the underlying partition.

Our emphasis is of course on multivariate splines, but it is instructive briefly to look at univariate splines for comparison. In one variable, $k = 1$, an interval $\Omega = [a, b]$ is partitioned into subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, \dots, N$, where $a = x_0 < x_1 < \dots < x_N = b$. We denote the partition by $\Delta = \{I_i, i = 1, \dots, N\}$. The relevant spline space is

$$S_d^r(\Delta) = \left\{s \in C^r(\Omega) : s|_I \in P_d, I \in \Delta\right\} \tag{1}$$

where P_d is the $(d+1)$-dimensional space of polynomials of degree d in one variable.

The set $S_d^r(\Delta)$ is a linear space and as such it has a dimension. In the case $k = 1$ it requires little more than first semester calculus to compute that dimension. Casually speaking, on the first interval, $[x_0, x_1]$ we have a polynomial of degree d with $d+1$ parameters. When we move to the next interval, we gain another $d+1$ parameters, but we also need to satisfy $r+1$ linear smoothness conditions (continuity of the spline and its first r derivatives) and thus we gain $d-r$ parameters. This

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http://dx.doi.org/10.1016/j.cagd.2015.11.007
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happens every time we move into a new interval, which we do $N-1$ times. Of course, one has to verify that the smoothness conditions are linearly independent, but that is a simple matter. Thus, in the univariate case,

$$\dim S_d^r(\Delta) = (d+1) + (N-1)(d-r) = N(d-r) + r + 1. \tag{2}$$

In two or more variables, ($k > 1$), various types of partitions are possible and have been investigated. The focus in this paper is on splines defined on *triangulations* of a polygonal region in $\mathbb{R}^2$, and, to a lesser degree, on splines defined on *tetrahedral decompositions* of a polyhedral domain in $\mathbb{R}^3$. Let T_i, $i = 1, 2, \ldots, N$ denote the triangles in a triangulation Δ of a region $\Omega \subset \mathbb{R}^2$.

Definition 1. A set $\Delta = \{T_1, \ldots, T_N\}$ of triangles in the plane is called a **triangulation** of $\Omega = \bigcup_{i=1}^N T_i$ provided that

1. If a pair of triangles in Δ intersect, then that intersection is either a common vertex or a common edge.
2. The domain Ω is homeomorphic to a disk.

We can now define our spline space of interest similarly as in the univariate case:

$$S_d^r(\Omega) = \left\{ s \in C^r(\Omega) : s\big|_{T_i} \in P_d, \quad i = 1, \ldots, N \right\} \tag{3}$$

where P_d is now the $(d+2)(d+1)/2$-dimensional space of polynomials of degree d in two variables. We will often write simply S_d^r instead of $S_d^r(\Delta)$ when this causes no confusion.

Note that in the univariate case the dimension of S_d^r, given in equation (2), depends only on the **number** of subintervals in the partition, and not on the **lengths** of those subintervals. For multivariate splines, the situation is very different. The dimension of S_d^r (and a number of other properties such as the solvability of certain interpolation problems) depends not only on the **combinatorics** (the numbers of interior and boundary vertices and edges, and triangles) and **topology** (the way triangles are connected) of the underlying partition, but also on its **geometry**, i.e., the precise location of the vertices. This fact is the source of the vastly increased complexity of multivariate spline spaces as compared to the simplicity of univariate spline spaces.

Throughout this paper for planar triangulations we use the notation

$$\begin{aligned} N &= \text{number of triangles} \\ V &= \text{number of vertices} \\ V_I &= \text{number of interior vertices} \\ V_B &= \text{number of boundary vertices} \\ E &= \text{number of edges} \\ E_I &= \text{number of interior edges} \\ E_B &= \text{number of boundary edges} \end{aligned} \tag{4}$$

The following *Euler relations* are useful:

$$V_B = E_B, \quad E_I = 3V_I + V_B - 3, \quad N = 2V_I + V_B - 2 \tag{5}$$

2. The B-form of a polynomial

A polynomial p of degree d in two variables x and y is usually written as

$$p(x, y) = \sum_{0 \le i+j \le d} \alpha_{ij} x^i y^j. \tag{6}$$

This form is focused on the origin and is inappropriate when dealing with a triangle that could be anywhere in the plane. An alternative is based on barycentric coordinates.

Definition 2. Let T be a non-degenerate triangle with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and let $\mathbf{x}$ be a point in $\mathbb{R}^2$. Then the **barycentric coordinates** b_1, b_2 and b_3 **of x with respect to** T are defined by

$$\mathbf{x} = \sum_{i=1}^3 b_i \mathbf{v}_i \qquad \text{and} \qquad \sum_{i=1}^3 b_i = 1. \tag{7}$$

Then it is easy to see that any bivariate polynomial p of degree d can be written uniquely in its *B-form* as

$$p(\mathbf{x}) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k. \tag{8}$$

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