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On regularity of generalized Hermite interpolation

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ABSTRACT

In this note we study the regularity of generalized Hermite interpolation and compare it to that of classical Hermite interpolation.
While every Hermite interpolation scheme is regular in one variable, the "classical Hermite interpolation schemes" in several variables are regular if and only if they are supported at one point. In this note we exhibit some regular generalized Hermite interpolation schemes supported at two points and study some limitation of existence of such schemes. The existence of such schemes provides a class of counterexamples to a conjecture of Jia and Sharma.

1. Introduction

In this note we will compare the regularity properties of generalized Hermite interpolation (as introduced in de Boor and Ron, 1991) to those of classical Hermite interpolation (cf. Lorentz and Lorentz, 1990; Lorentz, 2000). The classical Hermite interpolation schemes in several variables are regular if and only if they are supported at one point (cf. Lorentz, 2000). We show that this is still the case for generalized Hermite interpolation over the complex field and no longer the case over the real field. In particular, we study regular generalized Hermite interpolation schemes supported at two points. We also present some necessary conditions for the existence of such schemes. The constructed schemes provide a class of counterexamples to a conjecture made in Jia and Sharma (1991).

Some notations and terminology: The symbol $\mathbb{k}$ will stand for either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ and $\mathbb{Z}_+$ will denote the set of non-negative integers. We let $\mathbb{k}[\mathbf{x}] := \mathbb{k}[x_1, \ldots, x_d]$ be the algebra of polynomials in d variables with coefficients in $\mathbb{k}$. Thus every $p \in \mathbb{k}[\mathbf{x}]$ could be written as a finite sum

$$p(\mathbf{x}) = \sum_{\alpha} \hat{p}(\alpha)\mathbf{x}^{\alpha}$$

where $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, $\mathbf{x}^{\alpha} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and the coefficients $\hat{p}(\alpha) \in \mathbb{k}$. For every $p \in \mathbb{k}[\mathbf{x}]$ we use $\bar{p}$ to denote the complex conjugate of the polynomial p. Thus if $p(\mathbf{x}) = \sum \hat{p}(\alpha)\mathbf{x}^{\alpha}$ then $\bar{p}(\mathbf{x}) = \sum \overline{\hat{p}}(\alpha)\mathbf{x}^{\alpha}$ where $\bar{c}$ is the complex conjugate of c. We use D_j to denote the partial derivative with respect to x_j and for every $p \in \mathbb{k}[\mathbf{x}]$ the symbol $p(D)$ denotes the differential operator

$$p(D_1, \ldots, D_d) = \sum_{\alpha} \hat{p}(\alpha) D_1^{\alpha_1} \ldots D_d^{\alpha_d}$$

acting on $\mathbb{k}[\mathbf{x}]$. Finally, for $\mathbf{z} \in \mathbb{k}^d$ we use $\delta_{\mathbf{z}}$ to denote the point-evaluation functional: $\delta_{\mathbf{z}}(p) := p(\mathbf{z})$ for all $p \in \mathbb{k}[\mathbf{x}]$.

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A subspace $\mathcal{P} \subset \Bbbk[\mathbf{x}]$ is called D-invariant if $D_j p \in \mathcal{P}$ for every $p \in \mathcal{P}$ and $j = 1, \dots, d$. For any $F \subset \Bbbk[\mathbf{x}]$ we use $\mathcal{D}(F)$ to denote the least D-invariant subspace of $\Bbbk[\mathbf{x}]$ that contains F.

A generalized Hermite interpolation is determined by a finite-dimensional D-invariant subspace $\mathcal{P} \subset \Bbbk[\mathbf{x}]$, a finite set of D-invariant subspaces $\mathcal{P}_1, \dots, \mathcal{P}_s$ of $\mathcal{P}$ such that

$$\dim \mathcal{P} = \sum_{i=1}^{s} \dim \mathcal{P}_i \tag{1.1}$$

and a finite sequence of distinct points $\mathcal{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_s) \in (\Bbbk^d)^s$. We consider the problem of interpolation from the space $\mathcal{P}$ with interpolation conditions given by the direct sum

$$A(\mathcal{Z}, \mathcal{P}_j, j = 1, \dots, s) := \sum_{j=1}^{s} \{\delta_{\mathbf{z}_j} \circ \bar{p}(D) : p \in P_j\}, \tag{1.2}$$

i.e., for any $f \in \Bbbk[\mathbf{x}]$ we wish to find $p \in \mathcal{P}$ such that $\lambda(f) = \lambda(p)$ for all $\lambda \in A(\mathcal{Z}, P_j, j = 1, \dots, k)$.

In particular, if $\dim \mathcal{P}_j = 1$ for all j then (by D-invariance) each $\mathcal{P}_j$ is spanned by the constant function 1 and hence the Hermite interpolation problem becomes a Lagrange interpolation problem at the points $\mathcal{Z}$. If, on the other hand, $s = 1$ this means that we are interpolating various derivatives of a function at one point; we will refer to such interpolation as (generalized) Taylor interpolation.

We follow G. Lorentz (1989) (cf. also Jia and Sharma, 1991) and say that the interpolation scheme $\mathfrak{S} := \mathfrak{S}(\mathcal{P}, \mathcal{P}_1, \dots, \mathcal{P}_s)$ is *regular* if the interpolation problem from $\mathcal{P}$ with interpolation conditions $A(\mathcal{Z}, \mathcal{P}_j, j = 1, \dots, s)$ is uniquely solvable for any sequence of *pairwise distinct* points $\mathcal{Z}$ and any $f \in \Bbbk[\mathbf{x}]$. The interpolation scheme is *almost regular* if the interpolation problem is uniquely solvable for one (and therefore almost all) sequences of points $\mathcal{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_s)$ in $(\Bbbk^d)^s$. In the remaining case the interpolation scheme is *singular*.

Condition (1.1) assures that the number of interpolation conditions is the same as the dimension of the space $\mathcal{P}$ while the D-invariance of $\mathcal{P}_i$-s is a necessary and sufficient condition for the set

$$J := \{f \in \Bbbk[\mathbf{x}] : \lambda(f) = 0 \text{ for all } \lambda \in A(\mathcal{Z}, \mathcal{P}_j, j = 1, \dots, s)\}$$

to be a (zero-dimensional) ideal in $\Bbbk[\mathbf{x}]$.

In one variable this notion of Hermite interpolation coincides with the classical Hermite interpolation since the only finite-dimensional D-invariant subspaces of $\Bbbk[x]$ are polynomials of fixed maximal degree and are always regular. The classical Hermite interpolation problem in several variables (when all subspaces in question are polynomials of fixed degree) had been a subject of intensive study in recent years in Approximation theory (cf. Lorentz and Lorentz, 1990; Lorentz, 1992, 2000) as well as in algebraic geometry (cf. Alexander and Hirschowitz, 1995; Harris, 2011). In this article we will compare and contrast some of the results on regularity of classical multivariate Hermite interpolation problems to those of generalized ones.

One final note: choosing a basis $(p_1, \dots, p_N)$ for $\mathcal{P}$ and a basis $\lambda_1[\mathcal{Z}], \dots, \lambda_N[\mathcal{Z}]$ for $A(\mathcal{Z}, \mathcal{P}_j, j = 1, \dots, s)$ consider the generalized Vandermonde determinant

$$\Delta_{\mathfrak{S}}(\mathbf{z}_1, \dots, \mathbf{z}_s) := \det(\lambda_j[\mathcal{Z}](p_k))_{j,k=1}^{N}. \tag{1.3}$$

This is a polynomial in $d \times s$ variables:

$$(z_{1,1}, \dots, z_{1,d}, \dots, z_{s,1}, \dots, z_{s,d}) \in \mathbb{C}^{sd},$$

where

$$\mathbf{z}_j = (z_{j,1}, \dots, z_{j,d}), \ \ j = 1, \dots, s.$$

The regularity of a scheme $\mathfrak{S}(\mathcal{P}, \mathcal{P}_1, \dots, \mathcal{P}_s)$ is equivalent to the non-vanishing of this polynomial for all distinct $\mathbf{z}_1, \dots, \mathbf{z}_s$. The scheme $\mathfrak{S}$ is singular if and only if $\Delta_{\mathfrak{S}}$ is identically zero.

2. Regularity

It is a well-known fact that in one dimension the Hermite interpolation scheme is always regular while for $d > 1$ the classical Hermite interpolation scheme is regular if and only if it is a Taylor scheme, i.e., $\mathcal{Z}$ consists of one point. In this section we will show that this fact still holds for generalized Hermite interpolation in the complex setting ($\Bbbk = \mathbb{C}$) and fails in the real setting ($\Bbbk = \mathbb{R}$). The latter gives a counterexample to a conjecture of Jia and Sharma (cf. Jia and Sharma, 1991).

We will need the following rudimentary facts from algebraic geometry: A subset $\mathcal{V} \subset \Bbbk^N$ is called an affine variety if there exists a finite set of polynomials $f_1, \dots, f_s$ in $\Bbbk[x_1, \dots, x_N]$ such that

$$\mathcal{V} = \{\mathbf{x} \in \Bbbk^N : f_1(\mathbf{x}) = \dots = f_s(\mathbf{x}) = 0\};$$

the union of affine varieties is an affine variety (cf. Cox et al., 1997, p. 188, Theorem 15) and with every affine variety $\mathcal{V}$ we can associate the integer $\dim \mathcal{V}$ (cf. Cox et al., 1997, Chapter 9).