Computer Aided Geometric Design 45 (2016) 151–173

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

On the dimension of Tchebycheffian spline spaces over planar T-meshes

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ARTICLE INFO

Article history:
Available online 8 January 2016

Keywords:
Tchebycheffian splines
T-meshes
Dimension formula
Dimension bounds
Instability

ABSTRACT

In this paper we define Tchebycheffian spline spaces over planar T-meshes and we address the problem of determining their dimension. We extend to the Tchebycheffian spline context the homological approach previously used to characterize polynomial spline spaces over T-meshes, and we exploit this characterization in the study of the dimension. In particular, we give combinatorial lower and upper bounds for the dimension, and we show that these bounds coincide if the dimensions of the underlying extended Tchebycheff section spaces are large enough with respect to the smoothness, under some mild conditions on the T-mesh. Finally, we provide simple examples of Tchebycheffian spline spaces over T-meshes with unstable dimension, which means that their dimension depends on the exact geometry of the T-mesh. These results are extensions of those known in the literature for polynomial spline spaces over T-meshes.

1. Introduction

Tchebycheff spaces, or more precisely extended Tchebycheff spaces, are natural generalizations of algebraic polynomial spaces (Karlin and Studden, 1966; Schumaker, 2007). They are a popular tool in approximation theory, especially because they form a very flexible substitute for algebraic polynomial spaces to solve Hermite interpolation problems. Besides algebraic polynomial spaces, important examples of extended Tchebycheff spaces are the null spaces of differential operators with real constant coefficients.

Univariate Tchebycheffian splines are smooth piecewise functions with sections in extended Tchebycheff spaces. They have several advantages over classical (algebraic) polynomial splines, mainly due to the wide variety that extended Tchebycheff spaces offer. Despite this flexibility, many results of the polynomial framework extend in a natural way to the larger Tchebycheffian spline framework, ranging from approximation theory to geometric modelling, see Lyche et al. (1998), Mazure (2011), Schumaker (2007). In particular, Tchebycheffian splines admit a representation in terms of basis functions with similar properties as polynomial B-splines. Moreover, the elegant blossoming approach and classical algorithms (like degree elevation, knot insertion, differentiation formulas, etc.) can be rephrased for them (Goodman and Mazure, 2001; Lyche, 1985; Mazure, 2011).

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http://dx.doi.org/10.1016/j.cagd.2016.01.002

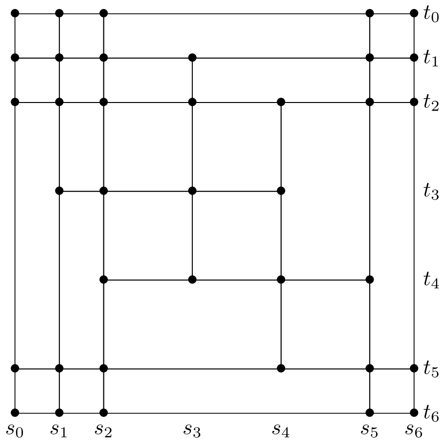

Fig. 1. An unstable T-mesh.

Multivariate extensions of Tchebycheffian splines can be easily obtained via the tensor-product approach and have been applied in different contexts. For example, tensor-products of so-called generalized splines (which are a special class of Tchebycheffian splines) are a promising problem-dependent tool in isogeometric analysis, a recent paradigm for the numerical treatment of partial differential equations (Manni et al., 2011).

Adaptive local refinement is important for both geometric modelling and numerical simulation. Unfortunately, a simple tensor-product spline structure lacks adequate local refinement. This triggered the interest in alternative spline structures supporting local refinement. Confining the discussion to local tensor-product structures, we mention (analysis-suitable) T-splines (Li and Scott, 2014; Sederberg et al., 2003), hierarchical splines (Forsey and Bartels, 1988; Giannelli et al., 2012), and locally refined (LR-)splines (Dokken et al., 2013). All of them can be seen as special cases of polynomial splines over T-meshes (Deng et al., 2006, 2008; Schumaker and Wang, 2012). In the more recent literature we also find some specific generalizations to the Tchebycheffian spline setting. For example, generalized T-splines (Bracco et al., 2014; Bracco and Cho, 2014), hierarchical generalized splines (Manni et al., 2014) and generalized splines on T-meshes (Bracco et al., 2016; Bracco and Roman, 2016) have been addressed. A multiresolution approach based on specific tensor-product Tchebycheffian splines has been considered in Lyche and Schumaker (2000). However, Tchebycheffian splines in their wide generality over T-meshes have not been previously investigated.

In this paper we consider Tchebycheffian spline spaces over T-meshes. As in the polynomial case, a complete understanding of these spline spaces requires the knowledge of the dimension of the spline space defined on a prescribed T-mesh for a given smoothness. Of course, it is of particular interest to understand when the dimension only depends on combinatorial quantities of the T-mesh (such as number of vertices, edges and faces), on the given smoothness, and on the componentwise dimensions (say $p_i + 1$) of the underlying extended Tchebycheff section spaces.

The dimension of spline spaces is said to be unstable if it depends on the exact geometry of the T-mesh. Spline spaces with unstable dimensions are not robust for practical use. Hence, it is important to detect whether or not there are instabilities in the dimension and to identify stable families of spaces. This instability phenomenon can be illustrated with the T-mesh in Fig. 1: the dimension of the C^1 quadratic polynomial spline space over the depicted T-mesh is 37 but reduces to 36 if, for example, the value s_3 is slightly perturbed (Li and Chen, 2011).

The dimension of polynomial spline spaces on a prescribed T-mesh for a given componentwise degree p_i and smoothness r_i has been addressed by several authors using different techniques, see Deng et al. (2006), Dokken et al. (2013), Li and Chen (2011), Mourrain (2014), Schumaker and Wang (2012), Zeng et al. (2015) and references therein, and it turns out to be a very challenging problem. Lower and upper bounds for the dimension are known, and an explicit expression has been determined in some special cases. In particular, the dimension is known for spline spaces over so-called quasi-cross-cut T-meshes (Mourrain, 2014) – these are meshes where each edge extends to the boundary – and for spline spaces with $p_i \geq 2r_i + 1$ under some mild conditions on the T-mesh (Deng et al., 2006; Mourrain, 2014; Schumaker and Wang, 2012). On the other hand, instability in the dimension can occur if the degree is not large enough with respect to the smoothness, see Li and Chen (2011) for the case $p_i = r_i + 1$ and see Berdinsky et al. (2012) for some specific examples with a larger gap between degree and smoothness.

The dimension problem of spline spaces over T-meshes faces the same difficulties as the dimension problem of polynomial spline spaces of total degree p over triangulations, see Lai and Schumaker (2007) and references therein. In the latter case, the dimension is known for spline spaces over quasi-cross-cut partitions (Chui and Wang, 1983; Schenk and Stillman, 1997), and for spline spaces with $p \geq 3r + 2$ (Hong, 1991; Ibrahim and Schumaker, 1991). Instability in the dimension has been illustrated for $p = 2r$ in Diener (1990). Some similar results are known for spline spaces of total degree p over general rectilinear partitions, see Chui and Wang (1983), Manni (1992) and references therein.