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The rational motion of minimal dual quaternion degree with prescribed trajectory ☆

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ABSTRACT

We give a constructive proof for the existence of a unique rational motion of minimal degree in the dual quaternion model of Euclidean displacements with a given rational parametric curve as trajectory. The minimal motion degree equals the trajectory's degree minus its circularity. Hence, it is lower than the degree of a trivial curvilinear translation for circular curves.

1. Introduction

A rational motion is a motion with only rational trajectories. In the dual quaternion model of SE(3), the group of rigid body displacements (Selig, 2005, Ch. 9), it is described by a rational curve on the Study quadric (Jüttler, 1993). In this article we construct a rational motion of minimal degree in the dual quaternion model with a given rational curve as trajectory, and we show that this motion is unique up to coordinate changes. This is an interesting result in its own right but it also has a certain potential for applications in computer graphics, computer aided design or mechanism science.

Usually, one defines the degree of a rational motion as the maximal degree of a trajectory (Jüttler, 1993). With this concept of motion degree, our problem becomes trivial as the curvilinear translation along the curve is already minimal. As we shall see, it is also minimal with respect to the dual quaternion degree if the prescribed trajectory is generic. The situation changes, however, if the trajectory is *circular*, that is, it intersects the absolute circle at infinity. In this case, the minimal achievable degree in the dual quaternion model is the curve degree minus half the number of conjugate complex intersection points with the absolute circle at infinity (the curve's *circularity*).

We will see that twice the circularity of a trajectory equals the trajectory degree minus the degree defect in the spherical component of the minimal motion. This leads to the rather strange observation that generic rational motions (without spherical degree defect) have very special (entirely circular) trajectories. Conversely, the minimal motion to generic (non-circular) curves are curvilinear translations which are special in the sense that their spherical degree defect is maximal.

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We continue this article with an introduction to the dual quaternion model of rigid body displacements in Section 2. There we also introduce motion polynomials and their relation to rational motions. Our results are formulated and proved in Section 3. The proof of the central result (Theorem 2) is constructive and can be used to actually compute the minimal rational motion by a variant of the Euclidean algorithm. We illustrate this procedure by two examples.

2. The dual quaternion model of rigid body displacements

In this article, we work in the dual quaternion model of the group of rigid body displacements. This model requires a minimal number of parameters while retaining a bilinear composition law. Moreover, it provides a rich algebraic and geometric structure. It is, for example, possible to use a variant of the Euclidean algorithm for computing the greatest common divisor (gcd) of two polynomials. This section presents the necessary theoretical background on dual quaternions.

2.1. Dual quaternions

The set $\mathbb{DH}$ of dual quaternions is an eight-dimensional associative algebra over the real numbers. It is generated by the base elements

$$1, \quad \mathbf{i}, \quad \mathbf{j}, \quad \mathbf{k}, \quad \varepsilon, \quad \varepsilon\mathbf{i}, \quad \varepsilon\mathbf{j}, \quad \varepsilon\mathbf{k}$$

and the non-commutative multiplication is determined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad \varepsilon^2 = 0, \quad \mathbf{i}\varepsilon = \varepsilon\mathbf{i}, \quad \mathbf{j}\varepsilon = \varepsilon\mathbf{j}, \quad \mathbf{k}\varepsilon = \varepsilon\mathbf{k}.$$

As important sub-algebras, the algebra of dual quaternions contains the real numbers $\mathbb{R} = \langle 1 \rangle$, the complex numbers $\mathbb{C} = \langle 1, \mathbf{i} \rangle$, the dual numbers $\mathbb{D} = \langle 1, \varepsilon \rangle$, and the quaternions $\mathbb{H} = \langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ (angled brackets denote a linear span). A dual quaternion may be written as $h = p + \varepsilon q$ where $p, q \in \mathbb{H}$ are quaternions. The conjugate dual quaternion is $\overline{h} = \overline{p} + \varepsilon\overline{q}$ and conjugation of quaternions is done by multiplying the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ with -1. It can readily be verified that the dual quaternion norm, defined as $\|h\| = h\overline{h}$, equals $p\overline{p} + \varepsilon(p\overline{q} + q\overline{p})$. It is a dual number. The non-invertible dual quaternions $h = p + \varepsilon q$ are precisely those with vanishing primal part $p = 0$.

An important application of dual quaternions is the modelling of rigid body displacements. The group of dual quaternions of unit norm modulo $\{\pm 1\}$ is isomorphic to SE(3), the group of rigid body displacements. A unit dual quaternion $h = p + \varepsilon q$ acts on a point x in the three dimensional real vector space $\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ according to

$$x \mapsto px\overline{p} + p\overline{q} - q\overline{p} = px\overline{p} + 2p\overline{q}. \tag{1}$$

Note that $-q\overline{p} = p\overline{q}$ because of the unit norm condition. It is convenient and customary to projectivise the space $\mathbb{DH}$ of dual quaternions, thus arriving at $\mathbb{P}^7$, the real projective space of dimension seven. Then, the unit norm condition can be relaxed to the non-vanishing of $p\overline{p}$ and the vanishing of $p\overline{q} + q\overline{p}$. In a geometric language, SE(3) is isomorphic to the points of a quadric $S \subset \mathbb{P}^7$, defined by $p\overline{q} + q\overline{p} = 0$, minus the points of a three-dimensional space, defined by $p = 0$. The quadric S is called the *Study quadric*. In this setting, the map (1) becomes

$$x \mapsto \frac{px\overline{p} + p\overline{q} - q\overline{p}}{p\overline{p}} = \frac{px\overline{p} + 2p\overline{q}}{p\overline{p}}.$$

The action of $h = p + \varepsilon q$ with $p \neq 0$, $p\overline{q} + q\overline{p} = 0$ can be extended to real projective three-space $\mathbb{P}^3$, modelled as projective space over $\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$. The point x represented by $x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is mapped according to

$$x \mapsto x_0 p\overline{p} + p(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})\overline{p} + 2x_0 p\overline{q} = px\overline{p} + 2x_0 p\overline{q}.$$

This is a convenient representation for studying rational curves as trajectories of rational motions.

2.2. Rational motions and motion polynomials

In the projective setting, a rational motion is simply a curve in the Study quadric S that admits a parameterisation by a polynomial

$$C = \sum_{i=0}^{n} c_i t^i \tag{2}$$

with dual quaternion coefficients $c_0, \ldots, c_n \in \mathbb{DH}$. The non-commutativity of $\mathbb{DH}$ necessitates some rules concerning notation and multiplication: Polynomial multiplication is defined by the convention that the indeterminate commutes with all coefficients, the ring of these polynomials in the indeterminate t is denoted by $\mathbb{DH}[t]$, the subring of polynomials with quaternion coefficients is $\mathbb{H}[t]$. We always write coefficients to the left of the indeterminate t. This convention is sometimes captured in the term “left polynomial” but we will just speak of a “polynomial”.