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A geometric method for computation of geodesic on parametric surfaces ☆

Peng Zhang, Ronglei Sun *, Tao Huang

School of Mechanical Science and Engineering, State Key Laboratory of Digital Manufacturing Equipment and Technology, Huazhong University of Science and Technology, Wuhan, 430074, China

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ABSTRACT

This paper proposes a geometric algorithm for computation of geodesic on surfaces. The geodesics on surfaces are traced in a simple way which is independent of the complex description of the geodesic equations. Through derivation process, the calculation error of this algorithm is obtained. A step size adjustment strategy which enables the step size adapt to the geometry of surface is introduced. The proposed method is also compared to some other well-known methods in this study. Many geodesics computed using these approaches on various B-spline surfaces or their equivalent tessellated surfaces have been presented. Experiments demonstrate that the proposed algorithm is efficient. Meanwhile, the results show that the step size adjustment strategy works well for most of the cases.

1. Introduction

Geodesic on a surface is an intrinsic geometric feature that plays an important role in a diversity of applications. The importance of geodesic is well known in the computer simulation of design and manufacturing process of composite components. For tape laying process, geodesic allows minimizing the steering of the tape (Debout et al., 2011). For filament winding process, it would be impossible to lay a filament in any way other than along a geodesic on a frictionless convex surface. Geodesic also finds its application in surface and shape processing, such as surface flattening (Azariadis and Aspragathos, 1997, 2001), segmentation, sampling, meshing and comparison of shapes (Peyré et al., 2010).

A curve on a surface is called a geodesic if and only if the normal vector to the curve is everywhere parallel to the normal vector of the surface. A geodesic can also be defined as a curve with zero geodesic curvature.

Available approaches for the computation of geodesic on surfaces can be classified broadly as analytical and numerical, and the later one is more widely used than the former one recently. The analytical approaches presented by Do Carmo (1976) are quite complex and closed form solutions cannot be found for geodesic on general surfaces. As for the numerical approaches, Beck et al. (1986) computed geodesic paths on a bicubic spline surface by using the fourth order Runge–Kutta method. Patrikalakis and Badris (1989) examined geodesic curves on parametric surfaces when they constructed offset curves on Rational B-spline surfaces. Sneyd and Peskin (1990) investigated the computation of geodesic paths on a generalized cylinder using a second order Runge–Kutta method. Hotz and Hagen (2000) presented a geometric method for the construction of geodesics on arbitrary surface. Their method is based on the fundamental property that geodesics are a

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* Corresponding author. Tel.: +86 27 87559416.
E-mail address: ronglei@mail.hust.edu.cn (R. Sun).

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generalization of straight lines on planes. The superiority of this method is that it makes us independent of the complex description of the surface. Ying and Candes proposed an approach for rapidly computing a very large number of geodesics on a smooth surface. Their approach is built upon the phase flow method and is especially well suited for the problem of computing creeping rays (Maekawa, 1996).

Finding the shortest path between two points on surface is a classical problem and it has many important applications. Maekawa introduced an approach for finding the shortest path between two points on a free-form parametric surface based on a relaxation method relying on finite difference discretization (Ying and Candes, 2006). Kasap et al. (2005) presented a numerical method for the computation of geodesic between two points on surface. They solve the nonlinear differential equations of geodesic via the finite-difference method and the iterative method. Chen (2010) proposed a geodesic-like curve that approaches to the geodesic on surface when the order of the curve reaches to infinity. Chen's geodesic-like curve has been proved to be accurate.

Many accurate discrete methods approach geodesics or the shortest paths on tessellated surfaces (Tucker, 1997; Ravi Kumar et al., 2003), polygonal surfaces (Polthier and Schmies, 1998; Kanai and Suzuki, 2001) and triangular meshes (Martinez et al., 2005; Surazhsky et al., 2005). Bose et al. (2011) did a survey which gave an overview of theoretically and practically relevant algorithms to compute geodesic paths and distances on three-dimensional surfaces. The algorithms are differentiated based on theoretical time complexity and approximation ratio.

In general, conventional approaches for computation of geodesic can be classified broadly into three types. First, solve the nonlinear differential equations of geodesic by using a numerical integration method such as Runge–Kutta method. Second, compute discrete geodesics on discrete surfaces by following some rules. Last, construct geodesics on smooth surfaces by using geometric methods. The first type is elegant and accurate, but the differential equations of geodesic are very complicated and generally not easy to solve. Discrete geodesics have been gaining attention as computer becomes increasingly more powerful and discretized models become more prevalent in geometric modeling. However, the discrete geodesics cannot be computed directly on the original smooth surface. There is little work on constructing geodesics directly on smooth surfaces by using geometric methods. The geometric method presented by Hotz and Hagen is complicated as it needs to compute projection of point of tangent plane in each step. Ravi Kumar's method can be extended to estimate geodesics on regular surfaces (Chen, 2010). However, this method does not perform on regular surfaces directly. The discrete geodesic is computed at first and then projected to the regular surface. In this work, we present a compact geometric method for constructing geodesics directly on smooth surfaces. Firstly, principle of the method is described in detail. Then, calculation error of the method is provided through derivation process. Finally, the method is tested on a group of different surfaces. See Table 1 for used notations.

2. Principle of the algorithm

This work describes an efficient method which aims to trace geodesic on parametric surfaces. The presented approach is independent of the complex description of the geodesic equations and is summarized as follows:

We look at a surface S which is given by a parameterization $\boldsymbol{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$, where x, y and z are differentiable functions of the parameters u and v. As this work focus on providing a computational method for the geodesic, we assume that S is regular.

A curve C lying entirely on the surface S can be expressed in parametric form by:

$$u = u(s), \qquad v = v(s), \tag{1}$$

where s is the arc length.

Let $Q_0 = \boldsymbol{r}(u_0, v_0)$ denote a starting point on curve C in three-dimensional space, and $\boldsymbol{T}_0$ the unit tangent vector of curve C at Q_0.

Then the three-dimensional directional vector $\boldsymbol{T}_0$ is pulled back to the parametric domain adopting a simple procedure (Piegl and Tiller, 1997), and we get the direction of C' at $q_0 = (u_0, v_0)$ which can be represented as $\boldsymbol{L}_0 = (u'(q_0), v'(q_0))$. Here C' and q_0 are the preimage of curve C and point Q_0 in the parametric domain, respectively.

Assume that $q_1 = (u_1, v_1)$ is the point lying in the neighborhood of q_0 on C', and it can be approximated as:

$$(u_1, v_1) = \big(u_0 + u'(q_0)\Delta s, v_0 + v'(q_0)\Delta s\big), \tag{2}$$

where Δs is arch length increment.

Let $Q_1 = \boldsymbol{r}(u_1, v_1)$ denote the three-dimensional image point of q_1 on S. According to the Frenet–Serret formulas, the unit tangent vector of curve C at Q_1 can be approximated as:

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 + k_0\boldsymbol{\beta}_0\Delta s, \tag{3}$$

where $\boldsymbol{T}_1$ is the unit tangent vector of curve C at Q_1, k_0 is the curvature of curve C at Q_0 and $\boldsymbol{\beta}_0$ is the principal normal vector of curve C at Q_0.

Suppose curve C is geodesic on surface S, and then Eq. (3) can be rewritten as:

$$\boldsymbol{T}_1 = \boldsymbol{T}_0 \pm k_0\boldsymbol{n}_0\Delta s, \tag{4}$$