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A rational cubic clipping method for computing real roots of a polynomial ☆

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A B S T R A C T

Many problems in computer aided geometric design and computer graphics can be turned into a root-finding problem of a polynomial equation. Among various solutions, clipping methods based on the Bernstein–Bézier form usually have good numerical stability. A traditional clipping method using polynomials of degree r can achieve a convergence rate of $r+1$ for a single root. It utilizes two polynomials of degree r to bound the given polynomial $f(t)$ of degree n, where $r = 2, 3$, and the roots of the bounding polynomials are used for clipping off the subintervals containing no roots of $f(t)$. This paper presents a rational cubic clipping method for finding the roots of a polynomial $f(t)$ within an interval. The bounding rational cubics can achieve an approximation order of 7 and the corresponding convergence rate for finding a single root is also 7. In addition, differently from the traditional cubic clipping method solving the two bounding polynomials in $O(n^2)$, the new method directly constructs the two rational cubics in $O(n)$ which can be used for bounding $f(t)$ in many cases. Some examples are provided to show the efficiency, the approximation effect and the convergence rate of the new method.

1. Introduction

Many problems in computer aided geometric design and computer graphics can be turned into a root-finding problem of polynomial equations, such as curve/surface intersection (Efremov et al., 2005; Liu et al., 2009; Nishita et al., 1990; Patrikalakis and Maekawa, 2002), point projection (Chen et al., 2008), collision detection (Choi et al., 2006; Lin and Gottschalk, 1998), and bisectors/medial axes computation (Elber and Kim, 2001). In principle, a system of polynomial equations of multiple variables can be turned into a univariate polynomial equation by using the resultant theory. This paper discusses the root-finding problem of a univariate polynomial equation within an interval.

Many references turn the given polynomial $f(t)$ into its power series, and a collection of related references can be found in McNamee (1993–2002), Isaacson and Keller (1966), Mourrain and Pavone (2005), Reuter et al. (2007), Rouillier and Zimmermann (2004). The Bernstein–Bézier form of $f(t)$ has a good numerical stability with respect to perturbations of the coefficients (Farouki et al., 1987; Farouki and Goodman, 1996; Jüttler, 1998). Several clipping methods based on the Bernstein–Bézier form are developed (Bartoň and Jüttler, 2007; Liu et al., 2009; Morken and Reimers, 2007; Sederberg and Nishita, 1990). Note that the number of zeros of a Bézier function is less or equal to that of its control polygon. The method

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in Morken and Reimers (2007) utilizes the corresponding control polygon to approximation $f(t)$, in which the zeros of the control polygon are used to approximate the zeros of $f(t)$ from one side. The method in Morken and Reimers (2007) achieves a convergence rate of 2 for a simple root. In principle, a B-spline (or Bézier) curve is bounded by the convex hull of its control polygon, the corresponding roots are then bounded by the roots of the convex hull. The corresponding approximation order of the approach using convex hull is 2 (Schulz, 2009). Comparing with the method in Morken and Reimers (2007), the r-clipping method in Bartoň and Jüttler (2007), Liu et al. (2009) bounds the zeros of $f(t)$ by using the zeros of two bounding polynomials of degree r, which achieves a higher approximation order $r+1$, where $r=2,3$.

In principle, one can also use rational polynomials to bound $f(t)$ for root finding. A rational quadratic polynomial has five free variables, which can achieve an approximation order 5 to $f(t)$. If two rational quadratics are utilized to bound $f(t)$, one can achieve a convergence rate of 5 for a simple root, which is much higher than that of 3 when using a quadratic clipping polynomial. However, in some cases when the curve $(t, f(t))$ is not convex within $[a,b]$, the denominators of the rational quadratic polynomials for bounding $f(t)$ may have one or more zeros within $[a,b]$, which leads to a bad approximation effect between $f(t)$ and its bounding polynomials.

A rational cubic polynomial, on the other hand, can approximate $f(t)$ in a much better way than that of a rational quadratic polynomial, even in case that $(t, f(t))$ is not convex within the given interval $[a,b]$. This paper presents a rational cubic clipping method which utilizes two rational cubic polynomials to bound $f(t)$ for root-finding. The bounding rational cubics achieve the approximation order 7 to $f(t)$ and the corresponding rational cubic clipping method can achieve a convergence rate of 7 for a simple root, which is much higher than that of 4 of previous cubic clipping methods. In addition, the method proposed in this paper directly constructs two rational cubic polynomials interpolating four positions and three derivatives of $f(t)$, which can bound $f(t)$ in many cases and it leads to a much higher computation efficiency. Some numerical examples are provided to show both higher convergence rate and higher computation efficiency of the new method.

The remainder of this paper is organized as follows. Section 2 provides an outline of clipping methods. Section 3 illustrates the rational cubic clipping method for finding two bounding rational cubics in details. Numerical examples and some further discussions are provided in Section 4, and the conclusions are drawn at the end of this paper.

2. Outline of the clipping methods

Suppose that $f(t)$, $t \in [a,b]$, is the given polynomial of degree n. The basic idea of the clipping methods is to find two bounding polynomials, and then to clip off the subintervals containing no roots of $f(t)$ by using the roots of the bounding polynomials. The clipping process continues until the lengths of the remaining subintervals are less than a given tolerance. Finally, the middle points of the remaining subintervals are recorded as the roots of $f(t)$.

The numerical convergence rate of a clipping method within an interval tends to be $\bar{m}/\bar{k}$, where $\bar{m}$ is the convergence rate of a method for a single root, and $\bar{k}$ is the sum of multiplicities of all of the roots within the interval. If $\bar{k}$ is large for some cases, the numerical convergence rate is very slow. In this paper, at the beginning, we apply the method in Morken and Reimers (2007) to divide the given interval into several sub-intervals by utilizing the zeros of the control polygon of the given Bézier function, which can improve the corresponding convergence rate.

2.1. The algorithm of a clipping step of the clipping method

In each clipping step, one needs to find two bounding polynomials for a given interval $[a,b]$ and then to split the interval $[a,b]$ into several subintervals by using the roots of the bounding polynomials. The subintervals containing no root of $f(t)$ are further clipped off. The computation of the two bounding polynomials is one of the key issues of a clipping step. Different clipping methods may obtain different bounding polynomials. Suppose that the two bounding polynomials are obtained such that $g_1(t) \le f(t) \le g_2(t)$. Note that the roots of $g_i(t)$ is usually easily obtained, it is trivial to check whether or not a root of $g_i(t)$ is a root of $f(t)$. Let Λ be a subinterval of $[a,b]$. We utilize the following lemma to clip the subintervals containing no roots of $f(t)$.

Lemma 1. *If $g_1(t) > 0$ or $g_2(t) < 0$ for all $t \in \Lambda$, then Λ can be removed.*

Proof. Firstly, if $g_1(t) > 0$, we have $f(t) \ge g_1(t) > 0$, for all $t \in \Lambda$. That is to say, Λ contains no root of $f(t)$ and can be removed.

Secondly, if $g_2(t) < 0$, we have $f(t) \le g_2(t) < 0$. Similarly, Λ contains no root of $f(t)$ and can also be removed. □

In this paper, we provide a rational cubic clipping method to find the roots of $f(t)$, in which two rational cubic polynomials are used for bounding $f(t)$. The roots of the two bounding rational cubic polynomials within an interval can be solved by using the Cardano formula (see more details in Liu et al., 2009). Finally, from Lemma 1, the roots of the bounding polynomials can be used for clipping the subintervals containing no roots of $f(t)$.