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On the monotonicity of generalized barycentric coordinates on convex polygons ☆

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ABSTRACT

We show that four well-known kinds of generalized barycentric coordinates in convex polygons share a simple monotonicity property: the coordinate function associated with a vertex is increasing along any line from the polygon boundary to that vertex. This shows that the coordinate functions have no local extrema and that their contours are single curves connecting pairs of points on the two edges adjacent to the vertex.

1. Introduction

Let $P \subset \mathbb{R}^2$ be a convex polygon, with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, $n \geq 3$, in some anticlockwise ordering. Fig. 1 shows an example with $n = 5$. Any functions $\phi_i : P \to \mathbb{R}$, $i = 1, \dots, n$, are *generalized barycentric coordinates* (GBCs) if, for all $\mathbf{x} \in P$, $\phi_i(\mathbf{x}) \geq 0$, $i = 1, \dots, n$, and

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1, \qquad \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}. \tag{1}$$

From this definition one can show that all GBCs have the same values on the boundary of the polygon, ∂P. Specifically, $\phi_i|_{\partial P} = f_i$, where the boundary function $f_i : \partial P \to \mathbb{R}$ has the values

$$\begin{aligned} f_i|_{e_j} &= 0, & j &\neq i-1, 1, \\ f_i((1-\mu)\mathbf{v}_{i\pm 1} + \mu \mathbf{v}_i) &= \mu, & \mu &\in [0, 1], \end{aligned} \tag{2}$$

and e_j is the j-th edge, $e_j := [\mathbf{v}_j, \mathbf{v}_{j+1}]$. Here and throughout, vertices, edges, and so on are indexed cyclically, i.e., $\mathbf{v}_{n+1} := \mathbf{v}_1$ etc.

We can see that ϕ_i is increasing along the edges e_{i-1} and e_i in the direction towards $\mathbf{v}_i$. In this note we show that several well known GBCs share a more general monotonicity property. We will say that $\phi_1, \dots, \phi_n$ are *monotonic* if, for all

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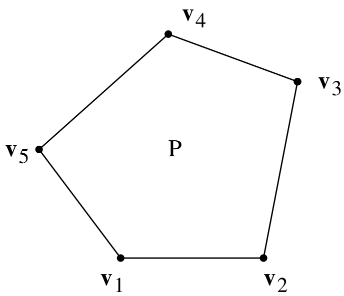

Fig. 1. Convex polygon.

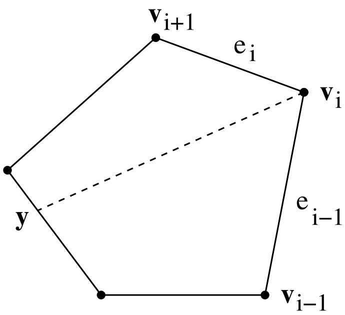

Fig. 2. Line segment connecting a boundary point $\mathbf{y}$ to $\mathbf{v}_i$.

$i = 1, \ldots, n$, and for all $\mathbf{y} \in \partial P$, $\mathbf{y} \neq \mathbf{v}_i$, the coordinate ϕ_i is increasing along the line segment from $\mathbf{y}$ to $\mathbf{v}_i$. Fig. 2 shows an example. Thus, by monotonic we mean that if

$$\mathbf{x} = (1-\lambda)\mathbf{y} + \lambda\mathbf{v}_i \qquad \text{and} \qquad \tilde{\mathbf{x}} = (1-\tilde{\lambda})\mathbf{y} + \tilde{\lambda}\mathbf{v}_i, \tag{3}$$

with $0 < \lambda < \tilde{\lambda} < 1$, then $\phi_i(\mathbf{x}) < \phi_i(\tilde{\mathbf{x}})$, and we need only check this property for boundary points $\mathbf{y} \notin e_{i-1}, e_i$. We will show that four distinct kinds of GBCs are monotonic: Wachspress coordinates, harmonic coordinates, Gordon–Wixom coordinates, and mean value coordinates.

Monotonicity does not follow from the definition (1) alone, since it is local to each point $\mathbf{x}$. We could construct an example of 'strange' GBCs that are neither continuous nor monotonic. If both coordinates of $\mathbf{x}$ are rational numbers, let $\phi_1(\mathbf{x}), \ldots, \phi_n(\mathbf{x})$ be the Wachspress coordinates of $\mathbf{x}$. Otherwise, if either coordinate is non-rational, let $\phi_1(\mathbf{x}), \ldots, \phi_n(\mathbf{x})$ be the mean value coordinates of $\mathbf{x}$. This defines a set of GBCs $\phi_1, \ldots, \phi_n$ for all $\mathbf{x}$ in P that are clearly not continuous (except on the boundary of P), and clearly not monotonic either.

The monotonicity property shows that each contour of ϕ_i is a single curve connecting a point on the edge e_{i-1} to the corresponding point on the edge e_i. Despite the fact that numerous contour plots of GBCs have appeared in the literature over several years, and they tend to exhibit this behavior, a mathematical proof seems to be missing. Another consequence of monotonicity is that ϕ_i cannot have any extrema in P.

It seems reasonable to use the simple term 'monotonic' because (a) this is the property that ϕ_i has along each line passing through $\mathbf{v}_i$ and (b) it seems unlikely that ϕ_i will be monotonic, in general, along any other line. Certainly, if we take any two points $\mathbf{y}_1$ and $\mathbf{y}_2$ on the boundary of P that lie on distinct edges other than e_i and e_{i-1}, then $\phi_i(\mathbf{y}_1) = \phi_i(\mathbf{y}_2) = 0$ and ϕ_i is positive on the line between $\mathbf{y}_1$ and $\mathbf{y}_2$, from which we conclude that ϕ_i is not monotonic on that line.

2. Wachspress' rational coordinates

To define Wachspress' coordinates (Wachspress, 1975; Warren, 1996; Meyer et al., 2002; Warren et al., 2007), let $\mathbf{n}_j$ denote the outward unit normal to the edge e_j, and for $\mathbf{x} \in P$, let $h_j(\mathbf{x})$ be the perpendicular distance of $\mathbf{x}$ to e_j, i.e., by the scalar product,

$$h_j(\mathbf{x}) = (\mathbf{v}_j - \mathbf{x}) \cdot \mathbf{n}_j = (\mathbf{v}_{j+1} - \mathbf{x}) \cdot \mathbf{n}_j. \tag{4}$$

Then the i-th Wachspress coordinate is $\phi_i = w_i/W$, where

$$w_i(\mathbf{x}) = \frac{c_i}{h_{i-1}(\mathbf{x})h_i(\mathbf{x})}, \qquad c_i = \mathbf{n}_{i-1} \times \mathbf{n}_i, \qquad W = \sum_{j=1}^{n} w_j,$$

and $\times$ is the scalar-valued cross product, so $\mathbf{n}_{i-1} \times \mathbf{n}_i = \det[\mathbf{n}_{i-1}, \mathbf{n}_i]$.