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Short Communication

Rate-distortion function for α-stable sources

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ABSTRACT

In this paper, we develop a numerical approximation based on the Blahut–Arimoto algorithm to the rate distortion function of sources with α-stable distribution both for the symmetric and the skewed cases. The calculated rate-distortion function provides bounds for lossy source coding/data compression and the achievable rates for a given distortion.

1. Introduction

Heavy-tailed or impulsive phenomena abound in many real life signal processing applications. Many man made signals such as web teletraffic or web transmission times [1], email based communications, and timing of individual human actions [2], SAR images of urban areas [3], or natural signals such as astronomical images [4] demonstrate impulsive characteristics.

In particular, it has been observed by various researchers that the wavelet coefficients of audio [6], and various types of images show heavy-tailed characteristics and various models such as Laplace distribution [5], generalized Gaussian distributions [7], Cauchy distribution [8] and α-stable distributions [10,11] have been proposed.

These observations seem to be contradicting the central limit theorem which indicates the Gaussian distribution for processes formed as the summation of a large number of variables. However, relaxing the condition of finite variance in the classical central limit theorem, the generalised version of the central limit theorem states that the sum of a large number of random variables with power-law tail distributions decreasing as $1/|x|^{\alpha+1}$ where $0 < \alpha < 2$ (and therefore having infinite variance) will tend to a stable distribution $S_\alpha(x)$ as the number of variables approaches infinity [9].

This theoretical justification has provided the α-stable distributions with wide acceptance as impulsive data models which is also supported experimentally by works such as, [10] where Achim et al. show its superiority to Laplace distribution in modelling the wavelet coefficients of biomedical ultrasound images and in [11] for SAR images. Recently, with the increasing popularity of compressed sensing and the need for efficient modelling of sparse data, α-stable distributions have been used in the framework of compressed sensing in works such as [12–14].

Despite the potential of α-stable distributions in modelling various types of impulsive and skewed signals and noise, studies in the coding and information theory on α-stable distributions have been very limited. Coding theory studies two fundamental problems: (i) reliable transmission of information over noisy channels via error–correction coding or channel coding (ii) the compression of signals to reduce storage or transmission (rate) requirements, that is, source coding. An important aspect of the first problem was addressed in the companion paper [15] where we have provided a computational method for calculating the capacity of a communication channel with α-stable noise. The current paper is on the dual problem of efficiently coding/compressing an α-stable distributed source.

Unfortunately, to the best of our knowledge, there is no work on the source coding properties of the α-stable distribution and its rate-distortion function, other than in [8] where Tsakalides and Nikias indicated a number of open problems for the study of source coding systems for heavy-tailed distributions.

It is, therefore, of fundamental importance to develop the rate-distortion function of this distribution family which would provide us with bounds in lossy and lossless source coding and insight into the limit of success of compressed sensing schemes with α-stable data models. In particular, the rate-distortion function can help

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us with the design of more realistic source coders and compressed sensing schemes. In this paper, we develop a numerical approximation to the rate-distortion function for α-stable distributions.

2. α-stable distribution

The α-stable distributions have received great interest in the last decade due to their success in modelling data which are too impulsive to be accommodated by the Gaussian distribution. The α-stable distribution family was known since early 20th century and it became a popular model for various types of signals.

The α-stable distribution is a generalization of the Gaussian distribution which can model also impulsive and skewed characteristics. This family of distributions is most conveniently defined by their characteristic functions [9]:

$$\phi(t) = \exp[i\delta t - |\gamma t|^{\alpha} B_{t,\alpha}] \tag{1}$$

$$B_{t,\alpha} = \begin{cases} 1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right) & \text{if} \quad \alpha \neq 1 \\ 1 + i\beta \operatorname{sgn}(t) \frac{2}{\pi} \log|t| & \text{if} \quad \alpha = 1 \end{cases} \tag{2}$$

where $\alpha \in (0, 2], \beta \in [-1, 1], \gamma \in (0, \infty), \delta \in (-\infty, \infty)$. α is the characteristic exponent and is the degree of impulsiveness of the distribution (Fig. 2). The smaller the value of α, the more impulsive the distribution is. For $\alpha = 2$, the distribution corresponds to the Gaussian distribution and $(\alpha, \beta) = (1, 0)$ to the Cauchy distribution. β is the symmetry parameter and sets the level of the skewness of the distribution (Fig. 3). $\beta = 0$ implies that the distribution is symmetric. γ is the scale parameter, which measures the spread of the samples around the mean (Fig. 4) similar to the variance for the Gaussian distribution. δ is the location parameter similar to the mean for the Gaussian distribution.

3. Rate-distortion function for α-stable distribution

The rate-distortion theory [16] provides achievable bounds on the performance of source coding methods. This bound is often

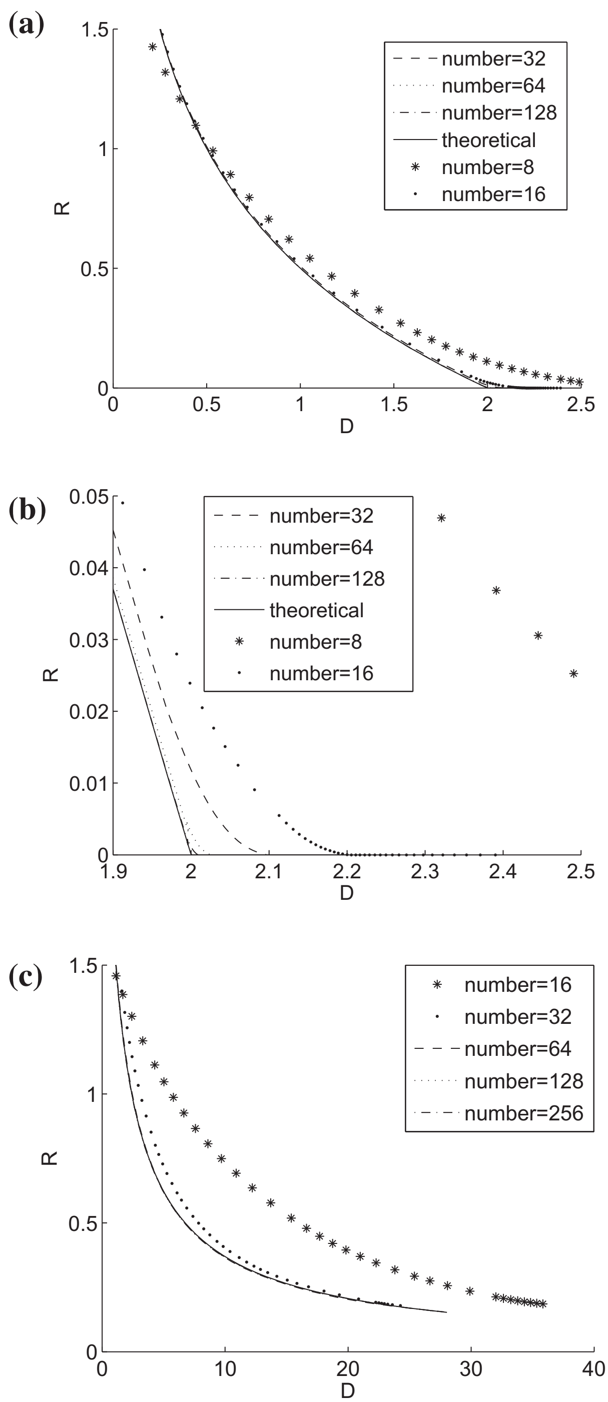

Fig. 1. (a) $R(D)$s calculated using BA algorithm for the Gaussian distribution ($\alpha = 2, \beta = 0, \gamma = 1, \delta = 0$), with varying number of quantization levels. (b) Zooming in to show the detail of (a). (c) $R(D)$ functions calculated using BA algorithm for the Cauchy distribution ($\alpha = 1.0, \beta = 0, \gamma = 1, \delta = 0$), with varying sample numbers.

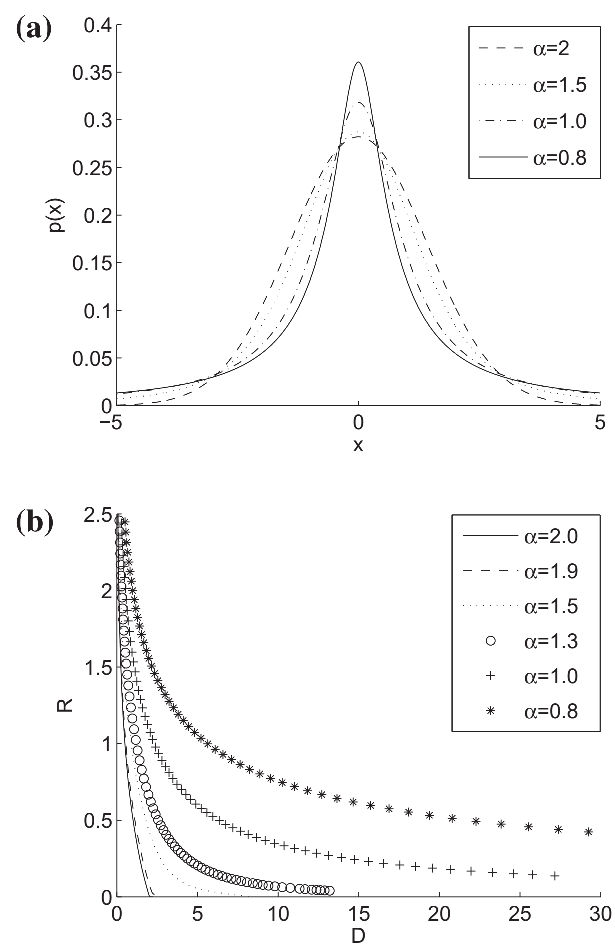

Fig. 2. (a) Probability density function for the α-stable distribution ($\beta = 0, \gamma = 1, \delta = 0$), with varying α. (b) $R(D)$ functions calculated using BA algorithm for these α-stable distributions.