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Discontinuous Galerkin methods for dispersive shallow water models in closed basins: Spurious eddies and their removal using curved boundary methods

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ABSTRACT

Discontinuous Galerkin methods offer a promising methodology for treating nearly hyperbolic systems such as dispersion-modified shallow water equations in complicated basins. Use of straight-edged triangular elements can lead to the generation of spurious eddies when wave fronts propagate around sharp, re-entrant obstacles such as headlands. While these eddies may be removed by adding strong artificial dissipation (e.g., eddy viscosity), for nearly inviscid simulations that focus on wave phenomena this approach is not reasonable. We demonstrate that the moderate order Discontinuous Galerkin methodology may be extended to curved triangular elements provided that the integral formulations are computed with high-order quadrature and cubature rules. Simulations with the new technique do not exhibit spurious eddy generation in idealized complex domains or real-world basins as exemplified by Pinehurst Lake, Alberta, Canada.

1. Introduction

Understanding the physical processes in lakes is of fundamental importance in a vast array of applications, ranging from water quality management to bio-geochemical cycling. Numerical modelling is a very valuable tool for improving and extending the current understanding of lake dynamics. A key challenge in the numerical modelling of lakes lies with representing the inherently complex geometry of coastlines. The irregularity of coastlines often motivates the modeller to choose a low-order method with a compact stencil of grid-points to approximate derivatives. Although high-order spectral and pseudospectral methods may thus appear inappropriate for modelling of real-world lakes due to their global stencil, these methods are quite powerful in simple geometries that represent idealized basins. The solution of a weakly non-hydrostatic single-layer model in periodic and annular domains with the high-order Fourier and Chebyshev pseudospectral methods has been recently explored by Steinmoeller et al. (2012, 2013). The methods developed in these works allow for the numerical modeling of circular or channel–like basins. While circular basins and channel–like basins may seem to be esoteric cases, they form a well studied class of problems in physical limnology dating back over a century (Thomson, 1872; Stocker and Imberger, 2003). High order numerical methods for such basins allow the robustness of classical solutions to be explored without the uncertainty associated with the inherent dissipation in many low order methods. This, in turn, allows for a rational set of hypotheses to be formulated for subsequent testing against field data. Regardless of these advances, the need for high-order methods that can capture more general geometries has been clear for some time and specific reasons for this need were identified in Steinmoeller et al. (2013).

Since a lake's coastal boundary generally specifies a physical domain with complex/irregular boundaries, the pseudospectral methods presented in Steinmoeller et al. (2012, 2013) are not sufficient for modelling real-world lakes. To represent more general geometries, we turn to the Discontinuous Galerkin finite element method (DG-FEM) as a high-order alternative to the low-order finite volume and finite element methods that are commonly used for irregular geometries. The results presented in this manuscript typically use local polynomial orders between $N = 4$ and 8. The methods are thus high-order in contrast to traditional finite element methods that typically use piece-wise linear or quadratic basis functions. See for instance Walkley (1999), who solved a Boussinesq-type system with a low-order finite element method (FEM). Low-order numerical work with Boussinesq-type systems applied in process studies has been carried out by Tomasson and Melville (1992), Brandt et al. (1997) and de la Fuente et al. (2008), for example.

It is worth stressing that the high-order DG-FEM is not the same as the spectral element method (SEM) (see Karniadakis and Sherwin, 2005) that represents the high-order extension of the traditional FEM. Both FEM and SEM are continuous Galkerin for-

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mulations which require C^0 continuity at element interfaces. Although DG-FEM and SEM both use a high-order orthogonal polynomial basis, the DG-FEM only imposes continuity in a weak sense through the specification of a numerical flux function at element edges in order to allow for stable advection schemes (Cockburn and Shu, 1989; Cockburn et al., 1990; Hesthaven and Warburton, 2008). The requirement of C^0 continuity in the SEM means that the method is not ideal for advection problems since an upwind-type scheme cannot be formulated to account for the preferred direction of propagation of information (Hesthaven and Warburton, 2008). In addition, modal filtering is known to have no effect at element interfaces where the basis polynomials are most oscillatory (Hesthaven et al., 2007). These shortcomings can lead to situations where Gibbs oscillations are trapped at element interfaces, as has been illustrated for the spectral element ocean model by Levin et al. (2006). However, it should be noted that modern treatments of FEM/SEM seek to overcome this shortcoming for advective problems by considering stabilization techniques such as the SUPG (streamline upwind/Petrov–Galerkin) method (Hughes, 1987) as well as the class of entropy-based viscosity methods (Nazarov et al., 2011). An alternative to the purely discontinuous approach has been recently proposed in the form of the hybridizable discontinuous Galerkin method that imposes strong continuity only in the edge-normal flux component (Rhebergen and Cockburn, 2012).

The specification of an upwind-biased numerical flux is usually furnished through the well-established theory of approximate Riemann solvers that are commonly used in the formulation of finite volume methods in order to propagate information between finite volume cells (see Toro, 1999 for an overview). It is for this reason that DG-FEM with piece-wise constant basis functions (order $N = 0$) is identical to the low-order finite volume method, as explained by Hesthaven and Warburton (2008).

In the following sections, we follow the techniques and developments for nodal DG-FEM presented by Hesthaven and Warburton (2008), building upon their techniques as necessary. We briefly explain the basic nodal DG-FEM formulation as the spatial discretization method for both hyperbolic and elliptic systems and the corresponding reduction to local operators in the context of a one-layer dispersive shallow water model. Following this, a comparison with the pseudospectral methods of Steinmoeller et al. (2012, 2013) is carried out as a means of validating the numerical scheme presented here and illustrating the resolution characteristics of the DG-FEM at varying polynomial orders. The necessity of curvilinear elements for general situations is illustrated by the formation of singular/spurious flow features that emerge because of the piece-wise linear representation of the boundary. It is then explained how the nodal DG-FEM method should be augmented with high-order cubature and quadrature integration rules to deal with the non-constant mapping Jacobians introduced by curvilinear elements.

In Section 2 we describe the basic numerical methods. Because standard techniques are used, many of the details are left to appendices. The modifications based on the use of curvilinear elements are described in Section 3. Results are presented in Section 4 which includes simulations on internal waves in a real-world lake: Pinehurst Lake, Alberta, Canada.

2. Methods

2.1. Governing equations

The governing equations for a single-layer reduced gravity model with non-hydrostatic corrections to the hydrostatic pressure (de la Fuente et al., 2008; Steinmoeller et al., 2012, 2013) are

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial (uh)}{\partial t} + \nabla \cdot ((uh)\mathbf{u}) = -g'h\frac{\partial \eta}{\partial x} + fvh + \frac{H^2}{6}\frac{\partial}{\partial x}\left(\nabla \cdot \frac{\partial (\mathbf{u}h)}{\partial t}\right), \tag{2}$$

$$\frac{\partial (vh)}{\partial t} + \nabla \cdot ((vh)\mathbf{u}) = -g'h\frac{\partial \eta}{\partial y} - fuh + \frac{H^2}{6}\frac{\partial}{\partial y}\left(\nabla \cdot \frac{\partial (\mathbf{u}h)}{\partial t}\right), \tag{3}$$

where $\mathbf{u}(x, y, t) = (u(x, y, t), v(x, y, t))$ is the velocity field, $h(x, y, t) = H(x, y) + \eta(x, y, t)$ is the total depth of the fluid below the pycnocline with H representing the undisturbed depth below the pycnocline, and η is the interfacial displacement from the undisturbed state. The constants g' and f are the reduced gravitational acceleration and the Coriolis frequency, respectively. These equations differ from the traditional hyperbolic shallow water model through the addition of the dispersive terms $\frac{H^2}{6}\nabla(\nabla \cdot (\mathbf{u}h)_t)$. The above system was developed for surface waves by Peregrine (1967) and used by Brandt et al. (1997) in their study of internal waves in the Strait of Messina. The model is for a thin layer of finite thickness below an infinitely deep layer and as such is only valid when the metalimnion is close to the bottom – a situation that occurs in the late summer, early autumn in many temperate lakes.

An efficient scheme for evolving the dispersive terms can be obtained by introducing the scalar auxiliary variable

$$z = \nabla \cdot (\mathbf{u}h)_t, \tag{4}$$

in order to reduce the momentum Eqs. (2) and (3) to a hyperbolic problem of the shallow water type plus the elliptic problem

$$\nabla \cdot \left(\frac{H^2}{6}\nabla z\right) - z = -\nabla \cdot \mathbf{a}, \tag{5}$$

that is referred to as a *wave continuity* equation by Eskilsson and Sherwin (2005). Here

$$\mathbf{a} = \begin{pmatrix} -\nabla \cdot ((uh)\mathbf{u}) - gh\eta_x + fvh \\ -\nabla \cdot ((vh)\mathbf{u}) - gh\eta_y - fuh \end{pmatrix}. \tag{6}$$

We have neglected bottom and surface stresses in Eqs. (1)–(3) since their inclusion into the numerical scheme is conceptually easy and contributes little to the discussion. We have also chosen to focus on the case of a single fluid layer of constant density, since the inclusion of multiple layers adds considerable complexity to numerical formulations that rely on approximate solutions to the corresponding nonlinear Riemann problem. See Mandli (2011) for a discussion on the two-layer Riemann problem in the context of the finite volume method.

2.2. Time-stepping technique

The time-stepping technique applied to the DG-FEM discretized version of the one-layer model closely follows the ‘scalar approach’ used for the pseudospectral discretization in Steinmoeller et al. (2012, 2013) and for the DG-FEM method in Eskilsson and Sherwin (2005) where splitting is applied such that advective and source terms are time-stepped first, followed by the dispersive terms. As in the works mentioned above, the time-stepping approach relies heavily on the ‘method of lines’ (see Leveque, 2007) where temporal and spatial discretizations are treated completely separately and a layer of abstraction may exist between these two discretizations.

Neglecting the dispersive terms for the time-being since they are not a part of the first splitting step, the method of lines can be applied by noticing that once the DG-FEM integral form