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Group extensions and graphs

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Abstract

A classical result of Gaschütz affirms that given a finite A-generated group G and a prime p, there exists a group $G^{\#}$ and an epimorphism $\varphi\colon G^{\#} \longrightarrow G$ whose kernel is an elementary abelian p-group which is universal among all groups satisfying this property. This Gaschütz universal extension has also been described in the mathematical literature with the help of the Cayley graph. We give an elementary and self-contained proof of the fact that this description corresponds to the Gaschütz universal extension. Our proof depends on another elementary proof of the Nielsen–Schreier theorem, which states that a subgroup of a free group is free.

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1. Introduction and statement of results

Recall that F is a *free group* over a set A when there exists a function $\iota\colon A \longrightarrow F$ such that given a group G and a map $f\colon A \longrightarrow G$, there exists a unique group homomorphism

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$\varphi\colon F \longrightarrow G$ such that $\iota\varphi = f$ (we will write maps and compositions on the right). The basic properties of the free group can be found, for instance, in [8, I, Section 19]. We will say that a group G is *generated* by a set A or *A-generated* if there exists a map $f\colon A \longrightarrow G$ such that the image of f is a generating system for G, that is, if there exists an epimorphism φ_G from the free group F over A onto G. In this case, we can identify the elements of A with their images in G and we will assume that φ_G is a fixed once given G and A. If G and H are two A-generated groups, with associated epimorphisms φ_G and φ_H, we will say that a homomorphism $\varphi\colon G \longrightarrow H$ *preserves generators* if $\varphi_G\varphi = \varphi_H$. Of course, a homomorphism between A-generated groups preserving generators must be an epimorphism.

The following classical result was proved by Gaschütz ([4], see also [2], Proposition β.3):

Theorem 1. *Given a finite A-generated group G and a prime p, then there exist an A-generated group $G^{\#}$ and an epimorphism $\varphi\colon G^{\#} \longrightarrow G$ preserving generators whose kernel is an elementary abelian p-group N with the following universal property:*

> *If H is an A-generated group and $\psi\colon H \longrightarrow G$ is an epimorphism preserving generators whose kernel is an elementary abelian p-group K, then there exists an epimorphism $\beta\colon G^{\#} \longrightarrow H$ such that $\beta\psi = \varphi$.*

If F is a free group over A and R is a normal subgroup of F such that $F/R \cong G$, then $G^{\#} = F/R'R^p$ satisfies the condition of Theorem 1. This group is used in the construction of a universal Frattini p-elementary extension of a finite group G, which is a group E with a normal, elementary abelian p-subgroup $A \neq 1$ such that $A \leq \Phi(E)$ and $E/A \cong G$ (see [2], B, Section 11 and Appendix β). This extension plays a major role in the proof of the theorem of Gaschütz, Lubeseder, and Schmid, which states that every saturated formation can be locally defined by a formation function ([5,6,9,13], see also [2, IV, Section 4] for details).

According to a theorem of Ribes and Zalesskiĭ [12], given finitely generated subgroups $H_1, \dots, H_n$ of a free group F, their product $H_1 \cdots H_n$ is closed in the profinite topology of F. This result confirmed a conjecture of Pin and Reutenauer [11] and completed a proof of the so-called *Rhodes type II conjecture* about finite monoids. The Ribes–Zalesskiĭ theorem has been proved and generalised in many ways. Connections of this theorem with other branches of Mathematics, like automata theory [16] and model theory [7], have also been investigated.

A constructive proof of the Ribes–Zalesskiĭ theorem was given by Auinger and Steinberg in [1]. The same proof can be easily adapted to give generalisations of this theorem for the pro-$\mathfrak{H}$ topology for varieties $\mathfrak{H}$ of groups satisfying that for each $G \in \mathfrak{H}$, there exists a prime p for which the wreath product $C_p \wr G$ is also in $\mathfrak{H}$. One of the ingredients of this proof is the construction of a group $G^{\mathbf{Ab}_p}$, for an A-generated finite group G and a prime p, based on the Cayley graph of the group G. The details of this construction will be presented later in Section 3. Elston [3], with the help of techniques from semigroup theory and category theory, proved that $G^{\mathbf{Ab}_p}$ satisfies the same universal property of $G^{\#}$ stated in Theorem 1. Consequently: