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Continuous functions that cut the real axis very often

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Abstract

We consider continuous functions $f: [0, 1] \to \mathbb{R}$ that cut the real axis at every point of a measurable set of positive measure and we construct examples where f fails to have bounded variation, and at the opposite end, where f admits derivatives of all orders.

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We say that a function $f: [0, 1] \to \mathbb{R}$ *cuts the real axis at a point* $x \in [0, 1]$ provided that $f(x) = 0$, and for every $\delta > 0$ there exist $x_1, x_2 \in [0, 1]$ such that $|x - x_i| < \delta$ for $1 \le i \le 2$ and $f(x_1) < 0 < f(x_2)$.

Recall that a function $f: [0, 1] \to \mathbb{R}$ is said to have *bounded variation* provided that

$$V_0^1(f) := \sup_P \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty,$$

where the supremum is taken over all partitions $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval $[0, 1]$. The above expression is called the *total variation of* f. It is a standard fact

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that any differentiable function f with continuous derivative f' has bounded variation and moreover,

$$V_0^1(f) = \int_0^1 |f'(x)|dx.$$

The purpose of this paper is to construct an example of a continuous function that cuts the real axis at every point of a measurable set of positive measure and that fails to have bounded variation, and at the opposite end, another example of a function that cuts the real axis at every point of a measurable set of positive measure and that admits derivatives of all orders.

Our first construction is based on the properties of the generalized Cantor sets. We follow the discussion of the generalized Cantor sets in the book of Folland [1, p. 40]. Let (ξ_n) be a sequence of positive numbers such that $\xi_0 = 1$ and $\xi_n > 2\xi_{n+1}$. Remove from $[0, 1]$ the open middle interval of length $\xi_0 - 2\xi_1$, obtaining a set B_1 that is the union of two disjoint closed intervals $[0, \xi_1]$ and $[1 - \xi_1, 1]$. Proceeding inductively, having constructed B_n, remove from each of its 2^n constituent intervals of length ξ_n the open middle interval of length $\xi_n - 2\xi_{n+1}$ to obtain a set B_{n+1} that is the union of 2^{n+1} disjoint closed intervals of length ξ_{n+1}. Finally, the intersection

$$B = \bigcap_{n\in\mathbb{N}} B_n$$

is called a *generalized Cantor set.* It is clear that $|B| = \lim 2^n \xi_n$. In particular, if $0 \le \alpha < 1$ and we take $\xi_n = \alpha 2^{-n} + (1-\alpha)3^{-n}$ then it is easy to check that the condition $\xi_n > 2\xi_{n+1}$ is satisfied and $|B| = \alpha$. From now on we restrict our attention to this particular choice.

Theorem 1. *Let $0 < \alpha < 1$ and let B be a generalized Cantor set with $|B| = \alpha$. There exists a continuous function $f : [0, 1] \to \mathbb{R}$ that fails to have bounded variation and that cuts the real axis at every $x \in B$.*

Proof. We construct a sequence of functions (f_n) that converges uniformly on $[0, 1]$ to the desired function. We proceed by induction. Let (c_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} c_n < \infty, \quad \text{and} \quad \lim_{n\to\infty} 2^n c_n = \infty$$

(for instance, the sequence $c_n = n/2^n$ does the job). First of all, define a function $h_1 : [0, 1 - 2\xi_1] \to \mathbb{R}$ by the expression $h_1(x) = c_1 \sin(2\pi x/(1 - 2\xi_1))$. Next, define a function $f_1 : [0, 1] \to \mathbb{R}$ by the expression

$$f_1(x) = \begin{cases} 0, & \text{if } 0 \le x \le \xi_1, \\ h_1(x - \xi_1), & \text{if } \xi_1 \le x \le 1 - \xi_1, \\ 0, & \text{if } 1 - \xi_1 \le x \le 1. \end{cases}$$

Suppose that we have constructed f_n and define a function $h_{n+1} : [0, \xi_n - 2\xi_{n+1}] \to \mathbb{R}$ by the expression

$$h_{n+1}(x) = c_{n+1} \sin\left(\frac{2\pi x}{\xi_n - 2\xi_{n+1}}\right).$$