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Numbers of points of surfaces in the projective 3-space over finite fields

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In the previous paper, we established an elementary bound for numbers of points of surfaces in the projective 3-space over $\mathbb{F}_q$. In this paper, we give the complete list of surfaces that attain the elementary bound. Precisely those surfaces are the hyperbolic surface, the nonsingular Hermitian surface, and the surface of minimum degree containing all $\mathbb{F}_q$-points of the 3-space.

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1. Introduction

Let S be a surface of degree d in $\mathbb{P}^3$ over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components, and $N_q(S)$ the number of $\mathbb{F}_q$-points of S. In the previous paper [8], we established the elementary bound for $N_q(S)$:

$$N_q(S) \le (d-1)q^2 + dq + 1, \tag{1}$$

and also gave three examples of surfaces that achieve the upper bound (1). The goal of this paper is to show that only those three are examples of such surfaces.

Theorem 1. *For a surface S in $\mathbb{P}^3$ over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components, if equality holds in (1), then the degree d of S is either 2 or $\sqrt{q}+1$ (when q is a square) or $q+1$. Furthermore, the surface S is projectively equivalent to one of the following surfaces over $\mathbb{F}_q$:*

(i) $X_0X_1 - X_2X_3 = 0$ *if* $d = 2$;
(ii) $X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = 0$ *if* $d = \sqrt{q}+1$;
(iii) $X_0X_1^q - X_0^qX_1 + X_2X_3^q - X_2^qX_3 = 0$ *if* $d = q+1$.

Notation 2. For an algebraic set X defined by equations over $\mathbb{F}_q$ in a projective space, the set of $\mathbb{F}_q$-points of X is denoted by $X(\mathbb{F}_q)$, and the cardinality of $X(\mathbb{F}_q)$ by $N_q(X)$. The symbol $\theta_q(r)$ denotes $N_q(\mathbb{P}^r)$, and we understand $\theta_q(0) = 1$.

The set of $\mathbb{F}_q$-planes of $\mathbb{P}^3$ is denoted by $\check{\mathbb{P}}^3(\mathbb{F}_q)$. For an $\mathbb{F}_q$-line l in $\mathbb{P}^3$, $\check{l}(\mathbb{F}_q)$ denotes the set $\{H \in \check{\mathbb{P}}^3(\mathbb{F}_q) \mid H \supset l\}$.

When Y is a finite set, ${}^{\#}Y$ denotes the cardinality of Y.

When M is a matrix, ${}^tM$ denotes the transposed matrix of M.

2. Review of some results in our previous works

2.1. Plane curves

To investigate plane sections of S, we need some results on plane curves.

Proposition 2.1 *(Sziklai bound). Let C be a curve of degree d in $\mathbb{P}^2$ over $\mathbb{F}_q$ without $\mathbb{F}_q$-line components. Then*

$$N_q(C) \le (d-1)q + 1 \tag{2}$$

unless C is the curve over $\mathbb{F}_4$ defined by

$$(X+Y+Z)^4 + (XY+YZ+ZX)^2 + XYZ(X+Y+Z) = 0 \tag{3}$$