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A bocs theoretic characterization of gendo-symmetric algebras

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ABSTRACT

Gendo-symmetric algebras were recently introduced by Fang and König in [7]. An algebra is called gendo-symmetric in case it is isomorphic to the endomorphism ring of a generator over a finite dimensional symmetric algebra. We show that a finite dimensional algebra A over a field K is gendo-symmetric if and only if there is a bocs-structure on $(A, D(A))$, where $D = Hom_K(-, K)$ is the natural duality. Assuming that A is gendo-symmetric, we show that the module category of the bocs $(A, D(A))$ is equivalent to the module category of the algebra eAe, when e is an idempotent such that eA is the unique minimal faithful projective-injective right A-module. We also prove some new results about gendo-symmetric algebras using the theory of bocses.

Introduction

A bocs is a generalization of the notion of coalgebra over a field. Bocses are also known under the name coring (see the book [4]). A famous application of bocses has been the proof of the tame and wild dichtomy theorem by Drozd for finite dimensional

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algebras over an algebraically closed field (see [5] and the book [3]). For any given bocs (A, W) over a finite dimensional algebra, one can define a corresponding module category and analyze it. Given a finite dimensional algebra A over a field K, it is an interesting question whether for a given A-bimodule W, there exists a bocs structure on (A, W). The easiest example to consider is the case $W = A$ and in this case the module category one gets is just the module category of the algebra A. Every finite dimensional algebra has a duality $D = Hom_K(-, K)$ and so the next example of an A-bimodule to consider is perhaps $W = D(A)$. We will characterize all finite dimensional algebras A such that there is a bocs structure on $(A, D(A))$ and find a surprising connection to a recently introduced class of algebras generalizing symmetric algebras (see [8]). Those algebras are called gendo-symmetric and are defined as endomorphism rings of generators of symmetric algebras. Alternatively these are the algebras A, where there exists an idempotent e such that eA is a minimal faithful injective-projective module and $D(Ae) \cong eA$ as (eAe, A)-bimodules. Then eAe is the symmetric algebra such that $A \cong End_{eAe}(M)$, for an eAe-module M that is a generator of mod-eAe. Famous examples of non-symmetric gendo-symmetric algebras are Schur algebras $S(n, r)$ with $n \geq r$ and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a complex semisimple Lie algebra (for a proof of this, using methods close to ours, see [11] and for applications see [9]). The first section provides the necessary background on bocses and algebras with dominant dimension larger than or equal to 2. The second section proves our main theorem:

A. Theorem *(Theorem 2.2). A finite dimensional algebra A is gendo-symmetric if and only if $(A, D(A))$ has a bocs-structure.*

We also provide some new structural results about gendo-symmetric algebras in this section. For example we show, using bocs-theoretic methods, that the tensor product over the field K of two gendo-symmetric algebras is again gendo-symmetric and we proof that $Hom_{A^e}(D(A), A)$ is isomorphic to the center of A, where A^e denotes the enveloping algebra of A.

In the final section, we describe the module category $\mathcal{B}$ of the bocs $(A, D(A))$ in case A is gendo-symmetric. The following is our second main result:

B. Theorem *(Theorem 3.3). Let A be a gendo-symmetric algebra with a minimal faithful projective-injective module eA. Then the module category of the bocs $(A, D(A))$ is equivalent to eAe-mod as K-linear categories.*

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