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The generalised nilradical of a Lie algebra

David A. Towers

Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, England, United Kingdom

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ABSTRACT

A solvable Lie algebra L has the property that its nilradical N contains its own centraliser. This is interesting because it gives a representation of L as a subalgebra of the derivation algebra of its nilradical with kernel equal to the centre of N. Here we consider several possible generalisations of the nilradical for which this property holds in any Lie algebra. Our main result states that for every Lie algebra L, $L/Z(N)$, where $Z(N)$ is the centre of the nilradical of L, is isomorphic to a subalgebra of $\mathrm{Der}(N^*)$ where N^* is an ideal of L such that N^*/N is the socle of a semisimple Lie algebra.

1. Introduction

Throughout, L will be a finite-dimensional Lie algebra, over a field F, with nilradical N and radical R. If L is solvable, then N has the property that $C_L(N) \subseteq N$. This property supplies a representation of L as a subalgebra of $\mathrm{Der}(N)$ with kernel $Z(N)$. The purpose

E-mail address: d.towers@lancaster.ac.uk.

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of this paper is to seek a larger ideal for which this property holds in all Lie algebras. The corresponding problem has been considered for groups (see, for example, Aschbacher [1, Chapter 11]). In group theory, the quasi-nilpotent radical (also called by some the generalised Fitting subgroup), $F^*(G)$, of a group G is defined to be $F(G) + E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the set of *components* of G: that is, the quasi-simple subnormal subgroups of the group. Also $F^*(G)/F(G)$ is equal to the socle of $C_G(F(G))F(G)/F(G)$. The generalised Fitting subgroup, $\tilde{F}(G)$, defined by $\tilde{F}(G)/\Phi(G)$ is the socle of $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G (see, for example, [7]). Here we consider various possible analogues for Lie algebras.

First we introduce some notation that will be used. The *centre* of L is $Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$; if S is a subalgebra of L, the *centraliser* of S in L is $C_L(S) = \{x \in L : [x, S] = 0\}$; the *Frattini ideal*, $\phi(L)$, of L is the largest ideal contained in all of the maximal subalgebras of L; we say that L is ϕ-free if $\phi(L) = 0$; the *socle* of S, $\operatorname{Soc} S$, is the sum of all of the minimal ideals of S; and the *L-socle* of S, $\operatorname{Soc}_L S$, is the sum of all of the minimal ideals of L contained in S. The symbol '$\oplus$' will be used to denote an algebra direct sum, whereas '$\dot{+}$' will denote a direct sum of the vector space structure alone.

We call L *quasi-simple* if $L^2 = L$ and $L/Z(L)$ is simple. Of course, over a field of characteristic zero a quasi-simple Lie algebra is simple, but that is not the case over fields of prime characteristic. For example, A_n where $n \equiv -1(modp)$ is quasi-simple, but not simple. This suggests using the quasi-simple subideals of a Lie algebra L to define a corresponding $E(L)$. However, first note that quasi-simple subideals of L are ideals of L. This follows from the following easy lemma.

Lemma 1.1. *If I is a perfect subideal (that is, $I^2 = I$) of L then I is a characteristic ideal of L.*

Proof. If I is perfect then $I = I^n$ for all $n \in \mathbb{N}$. It follows that $[L, I] = [L, I^n] \subseteq L\,(\operatorname{ad} I)^n \subseteq I$ for some $n \in \mathbb{N}$, and hence that I is an ideal of L. But now, if $D \in \operatorname{Der}(L)$, then $D([x_1, x_2]) = [x_1, D(x_2)] + [D(x_1), x_2] \in I$ for all $x_1, x_2 \in I$. Hence $D(I) = D(I^2) \subseteq I$. □

Combining this with the preceding remark we have the following.

Lemma 1.2. *Let L be a Lie algebra over a field of characteristic zero. Then I is a quasi-simple subideal of L if and only if it is a simple ideal of L.*

We say that an ideal A of L is *quasi-minimal* in L if $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Clearly a quasi-simple ideal is quasi-minimal. Over a field of characteristic zero, an ideal A of L is quasi-minimal if and only if it is simple. So an alternative is to define $E(L)$ to consist of the quasi-minimal ideals of L. We investigate these two possibilities in sections 3 and 5.