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Totally acyclic complexes ☆

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ABSTRACT

It is known that over an Iwanaga–Gorenstein ring the Gorenstein injective (Gorenstein projective, Gorenstein flat) modules are simply the cycles of acyclic complexes of injective (projective, flat) modules. We consider the question: are these characterizations *only* working over Iwanaga–Gorenstein rings? We prove that if R is a commutative noetherian ring of finite Krull dimension then the following are equivalent: 1. R is an Iwanaga–Gorenstein ring. 2. Every acyclic complex of injective modules is totally acyclic. 3. The cycles of every acyclic complex of Gorenstein injective modules are Gorenstein injective. 4. Every acyclic complex of projective modules is totally acyclic. 5. The cycles of every acyclic complex of Gorenstein projective modules are Gorenstein projective. 6. Every acyclic complex of flat modules is F-totally acyclic. 7. The cycles of every acyclic complex of Gorenstein flat modules are Gorenstein flat. Thus we improve slightly on a result of Iyengar and Krause; in [22] they proved that for a commutative noetherian ring R with a dualizing complex, the class of acyclic complexes

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of injectives coincides with that of totally acyclic complexes of injectives if and only if R is Gorenstein. We replace the dualizing complex hypothesis by the finiteness of the Krull dimension, and add more equivalent conditions.
In the second part of the paper we focus on the noncommutative case. We prove that for a two sided noetherian ring R of finite finitistic flat dimension that satisfies the Auslander condition the following are equivalent: 1. Every complex of injective (left and respectively right) R-modules is totally acyclic. 2. R is Iwanaga–Gorenstein.

1. Introduction

Homological algebra is at the root of modern techniques in many areas of mathematics including commutative and noncommutative algebra, algebraic geometry, algebraic topology and representation theory. Not only do these areas make use of the homological methods but homological algebra serves as a common language and this makes interactions between these areas possible and fruitful. A relative version of homological algebra is the area called Gorenstein homological algebra. This newer area started in the late 60s when Auslander introduced a class of finitely generated modules that have a complete resolution. Auslander used these modules to define the notion of the G-dimension of a finite module over a commutative noetherian local ring. Then Auslander and Bridger extended the definition to two sided noetherian rings (1969). The area really took off in the mid 90s, with the introduction of the Gorenstein (projective and injective) modules by Enochs and Jenda ([10]). Avramov, Buchweitz, Martsinkovsky, and Reiten proved that if the ring R is both right and left noetherian and if G is a finitely generated Gorenstein projective module, then Enochs' and Jenda's definition agrees with that of Auslander's and Bridger's of a module of G-dimension zero. The Gorenstein flat modules were introduced by Enochs, Jenda and Torrecillas as another extension of Auslander's Gorenstein dimension ([13]).

The Gorenstein homological methods have proved to be very useful in characterizing various classes of rings. Also, methods and results from Gorenstein homological algebra have successfully been used in algebraic geometry, as well as in representation theory. But the main problem in using the Gorenstein homological methods is that they can only be applied when the corresponding Gorenstein resolutions exist. So the main open problems in this area concern identifying the type of rings over which Gorenstein homological algebra works. Of course one hopes that this is the case for any ring. But so far only the existence of the Gorenstein flat resolutions was proved over arbitrary rings (in [30], 2014). The existence of the Gorenstein projective resolutions and the existence of the Gorenstein injective resolutions are still open problems. And they have been studied intensively in recent years (see for example [6,9,12,15,18,21,26]).