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An algorithm for computing weight multiplicities in irreducible modules for complex semisimple Lie algebras ☆

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ having rank l and let V be an irreducible finite-dimensional $\mathfrak{g}$-module having highest weight λ. Computations of weight multiplicities in V, usually based on Freudenthal's formula, are in general difficult to carry out in large ranks or for λ with large coefficients (in terms of the fundamental weights). In this paper, we first show that in some situations, these coefficients can be "lowered" in order to simplify the calculations. We then investigate how this can be used to improve the aforementioned formula of Freudenthal, leading to a more efficient version of the latter in terms of complexity as well as to a way of dealing with certain computations in unbounded ranks. We conclude by illustrating the last assertion with a concrete example.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Set $l = \operatorname{rank} \mathfrak{g}$ and fix an ordered base $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of the corresponding root system $\Phi = \Phi^+ \sqcup \Phi^-$ of $\mathfrak{g}$, where Φ^+ and Φ^- denote the sets of positive and negative roots of Φ, respectively. Also let $\lambda_1, \dots, \lambda_l$ denote the so-called fundamental weights corresponding to our choice of base Π and write $\Lambda = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_l$ for the associated integral weight lattice. Finally, let Λ^+ denote the set of dominant integral weights and recall the existence of a partial order on Λ, defined by $\mu \preccurlyeq \lambda$ if and only if $\lambda - \mu \in \Gamma$, where $\Gamma \subset \Lambda$ is the monoid of $\mathbb{Z}_{\geqslant 0}$-linear combinations of simple roots.

It is well-known that the set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules is in one-to-one correspondence with the set Λ^+ of dominant integral weights. Furthermore, a class representative $L(\lambda)$ corresponding to a given weight $\lambda \in \Lambda^+$ can be constructed as the quotient of the so-called *Verma module of weight* λ, written $\Delta(\lambda)$, by its unique maximal submodule $\operatorname{rad}(\lambda)$, that is

$$L(\lambda) = \Delta(\lambda)/\operatorname{rad}(\lambda).$$

Even though infinite-dimensional, Verma modules are $\mathfrak{h}$-semisimple, i.e., can be decomposed as direct sums of their weight spaces. Moreover, such decompositions are well understood: for a given dominant integral weight $\lambda \in \Lambda^+$ and any integral weight $\mu \in \Lambda$, a basis for the weight space in $\Delta(\lambda)$ corresponding to μ is known (see (7) in Section 2.4 below) and hence so is the multiplicity $\mathrm{m}_{\Delta(\lambda)}(\mu)$ of μ in $\Delta(\lambda)$. In addition, one gets that the set $\Lambda(\Delta(\lambda))$ of weights of $\Delta(\lambda)$ simply consists of all $\mu \in \Lambda$ such that $\mu \preccurlyeq \lambda$.

Unfortunately, not that much can be said about weight spaces in $L(\lambda)$ for an arbitrary dominant integral weight $\lambda \in \Lambda^+$. Firstly, finding out if a given weight $\mu \prec \lambda$ belongs to the set $\Lambda(\lambda)$ of weights of $L(\lambda)$ is far from being immediate, as it generally requires one to determine the unique dominant integral weight to which μ is conjugate (under the action of the Weyl group of $\mathfrak{g}$). Moreover, an explicit description of the (often very large) set

$$\Lambda^+(\lambda) = \Lambda(\lambda) \cap \Lambda^+$$

for $\lambda \in \Lambda^+$ with large coefficients (when written as a $\mathbb{Z}$-linear combination of fundamental weights) is usually hard to come by (see [11] for a recursive method). The first result in this paper shows that under certain assumptions on $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, the multiplicity of μ in $L(\lambda)$ is the same as the multiplicity of μ' in $L(\lambda')$, where λ' is a dominant integral weight whose coefficients (again, when written as a $\mathbb{Z}$-linear combination of fundamental weights) are smaller than or equal to those of λ, and $\mu' \in \Lambda$ is the unique integral weight satisfying $\lambda' - \mu' = \lambda - \mu$. The proof essentially relies on the existence of an explicit description of the maximal submodule $\operatorname{rad}(\lambda)$ of $\Delta(\lambda)$ (see [8, Section 2.6] or Theorem 2.4 below).