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The quaternion group has ghost number three

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We prove that the group algebra of the quaternion group Q_8 over any field of characteristic two has ghost number three.

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1. Introduction

The study of ghost maps in stable categories originated with Freyd's generating hypothesis in homotopy theory [6], which is still an open question. In this paper we are concerned with ghosts in modular representation theory. Let G be a group and K a field of characteristic p. A map $f\colon M \to N$ in the stable category $\operatorname{stmod}(KG)$ of finitely generated KG-modules is called a *ghost* if it vanishes under Tate cohomology, that is if $f_*\colon \hat{H}^*(G,M) \to \hat{H}^*(G,N)$ is zero. The ghost maps then form an ideal in $\operatorname{stmod}(KG)$, and Chebolu, Christensen and Mináč [3] define the *ghost number* of KG to be the nilpotency degree of this ideal.

Determining the exact value of the ghost number is hard in all but the simplest cases. In [4], Christensen and Wang studied ghost numbers for p-group algebras. They gave conjectural upper and lower bounds for the ghost number of an arbitrary p-group, and also showed that the ghost number (over a field of characteristic two) of the quaternion group Q_8 is either three or four. In our earlier paper [1], we established most cases of their conjectural bounds. In this paper, we shall prove the following theorem.

Theorem 1.1. *Let K be any field of characteristic two. Then the group algebra KQ_8 has ghost number three.*

Let us call a composition of n ghost maps an *n-fold ghost*. Given the result of Christensen and Wang on Q_8, our Theorem 1.1 is equivalent to the statement that every threefold ghost map $M \xrightarrow{f} N$ is stably trivial. To show this, we take any embedding $M \rightarrowtail I$ of M in a finitely generated KQ_8-module and show that f factors through I.

In Section 2, we recall Dade's presentation of the group algebra KQ_8 and derive some properties of ghost maps, including the crucial Lemma 2.5. In Section 3, we recall a theorem of Kronecker which classifies the linear relations on a vector space V. This leads us to the construction of the lift in Section 4: We have $I = KQ_8 \otimes_K V$ for some K-vector space V. As we may assume M to be projective-free, we have $M \subseteq J \otimes_K V$ for J the Jacobson radical $J = J(KQ_8)$. Since a threefold ghost kills $\operatorname{soc}^3(M)$, it follows that f factors through $M/\operatorname{soc}^3(M)$, which is a subspace of $(J/J^2) \otimes_K V \cong V^2$. That is, $M/\operatorname{soc}^3(M)$ is a linear relation on V; and using Lemma 2.5 we are able to construct a lift for each indecomposable summand in its Kronecker decomposition, thus proving the theorem.

2. Ghost maps and Dade's generators

We only need the following property of ghost maps.

Lemma 2.1 *([3], Proposition 2.1). Let G be a p-group, K a field of characteristic p, and $M \xrightarrow{f} N$ a ghost map between projective-free KG-modules. Then $\operatorname{Im}(f) \subseteq \operatorname{rad}(N)$ and $\operatorname{soc}(M) \subseteq \ker(f)$.* $\square$