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Products of commutators in a Lie nilpotent associative algebra

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ABSTRACT

Let F be a field and let $F\langle X\rangle$ be the free unital associative algebra over F freely generated by an infinite countable set $X = \{x_1, x_2, \ldots\}$. Define a left-normed commutator $[a_1, a_2, \ldots, a_n]$ recursively by $[a_1, a_2] = a_1a_2 - a_2a_1$, $[a_1, \ldots, a_{n-1}, a_n] = [[a_1, \ldots, a_{n-1}], a_n]$ $(n \geq 3)$. For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $F\langle X\rangle$ generated by all commutators $[a_1, a_2, \ldots, a_n]$ $(a_i \in F\langle X\rangle)$.

Let F be a field of characteristic 0. In 2008 Etingof, Kim and Ma conjectured that $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$ if and only if m or n is odd. In 2010 Bapat and Jordan confirmed the "if" direction of the conjecture: if at least one of the numbers m, n is odd then $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$. The aim of the present note is to confirm the "only if" direction of the conjecture. We prove that if $m = 2m'$ and $n = 2n'$ are even then $T^{(m)}T^{(n)} \not\subseteq T^{(m+n-1)}$. Our result is valid over any field F.

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1. Introduction

Let F be a field. Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set and let $F\langle X\rangle$ be the free associative algebra over F freely generated by X. Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ recursively by $[a_1, a_2] = a_1a_2 - a_2a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ $(n \geq 3)$. For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $F\langle X\rangle$ generated by all commutators $[a_1, a_2, \dots, a_n]$ $(a_i \in F\langle X\rangle)$.

In 2008 Etingof, Kim and Ma [9] made a conjecture (see Conjecture 3.6 in the arXiv version of [9]) that can be reformulated as follows:

Conjecture 1.1 *(see [9]). Let F be a field of characteristic 0. Then $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$ if and only if m or n is odd.*

In [9] this conjecture was confirmed for m and n such that $m + n \leq 7$. In 2010 Bapat and Jordan [2, Corollary 1.4] confirmed the "if" direction of the conjecture for arbitrary m, n.

Theorem 1.2 *(see [2]). Let F be a field of characteristic $\neq 2, 3$. Let $m, n \in \mathbb{Z}$, $m, n > 1$ and at least one of the numbers m, n is odd. Then*

$$T^{(m)}T^{(n)} \subset T^{(m+n-1)}. \tag{1}$$

The aim of the present note is to confirm the "only if" direction of the conjecture. Our main result is as follows.

Theorem 1.3. *Let F be a field and let $m = 2m'$, $n = 2n'$ be arbitrary positive even integers. Then*

$$T^{(m)}T^{(n)} \not\subseteq T^{(m+n-1)}.$$

Recall that an associative algebra A is Lie nilpotent of class at most c if $[u_1, \dots, u_c, u_{c+1}] = 0$ for all $u_i \in A$. We deduce Theorem 1.3 from the following result.

Theorem 1.4. *Let F be a field and let $m = 2m'$, $n = 2n'$ be arbitrary positive even integers. Then there exists a unital associative algebra A such that the following two conditions are satisfied:*

i) for all $u_1, u_2, \dots, u_{m+n-1} \in A$ we have

$$[u_1, u_2, \dots, u_{m+n-1}] = 0,$$

that is, the algebra A is Lie nilpotent of class at most $m + n - 2$;