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Locally nilpotent derivations and automorphism groups of certain Danielewski surfaces

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A B S T R A C T

We describe the set of all locally nilpotent derivations of the quotient ring $\mathbb{K}[X, Y, Z]/(f(X)Y - \varphi(X, Z))$ constructed from the defining equation $f(X)Y = \varphi(X, Z)$ of a *generalized Danielewski surface* in $\mathbb{K}^3$ for a specific choice of polynomials f and φ, with $\mathbb{K}$ an algebraically closed field of characteristic zero. As a consequence of this description we calculate the ML-invariant and the Derksen invariant of this ring. We also determine a set of generators for the group of $\mathbb{K}$-automorphisms of $\mathbb{K}[X, Y, Z]/(f(X)Y - \varphi(Z))$ also for a specific choice of polynomials f and φ.

Introduction

The term *Danielewski surface* usually refers to surfaces given by an equation of the form $X^n Z = P(Y)$, with $n \in \mathbb{N}$ and certain polynomials $P(Y) \in \mathbb{C}[Y]$, because such surfaces were studied by Danielewski in connection with the famous Cancellation Problem (see [11]). Its generalizations continue to be a source of interest for current research.

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For certain authors, a Danielewski surface is an affine surface which is algebraically isomorphic to a surface defined by an equation of the form $X^nZ - P(Y) = 0$, for (specific choice of) $P(Y) \in \mathbb{C}[Y]$, or even a surface defined by the equation $X^nZ - Q(X,Y) = 0$, where $Q(X,Y)$ is a polynomial satisfying certain properties. Over the past 30 years, many works on surfaces given by this type of equation were published under algebraic and algebraic–geometric approach (see [3,5]). Following we trace some papers which motivated this work:

- L. Makar-Limanov in [9,11] computed automorphism groups of surfaces in $\mathbb{C}^3$ defined by equation of the form $X^nZ - P(Y) = 0$, where $n \geq 1$ and $P(Y)$ is a nonzero polynomial. The ML-invariant is used in [11] to find the group of $\mathbb{K}$-automorphisms of the ring $\mathbb{K}[X,Y,Z]/(X^nZ - P(Y))$, where $n > 1$ and $\deg P \geq 2$.
- D. Daigle in [2] studied the locally nilpotent derivations of the quotient ring $R = \mathbb{K}[X,Y,Z]/(XY - \varphi(Z))$ and showed that certain subgroups of $\mathbb{K}$-automorphisms of R act transitively on the kernels of the nontrivial locally nilpotent derivations on R.
- A. Crachiola in [1] obtained similar results for slightly different surfaces defined by the equations $X^nZ - Y^2 - \sigma(X)Y = 0$, where $n \geq 2$ and $\sigma(0) \neq 0$, defined over arbitrary base field.
- A. Dubouloz and P.-M. Poloni [7] considered more general surfaces defined by equations $X^nZ - Q(X,Y) = 0$, where $n \geq 2$ and $Q(X,Y)$ is a polynomial with coefficients in an arbitrary base field such that $Q(0,Y)$ splits with $r \geq 2$ simple roots. This class contains most of the surfaces considered by L. Makar-Limanov, D. Daigle, and A. Crachiola.

In this paper we obtain some similar results for a class of Danielewski surfaces given by the equation $f(X)Y - \varphi(X,Z) = 0$, that means we study the ring

$$\mathcal{B} = \mathbb{K}[X,Y,Z]/(f(X)Y - \varphi(X,Z)),$$

where $\mathbb{K}$ is an algebraically closed field of characteristic zero, X, Y and Z are indeterminates over $\mathbb{K}$, $\varphi(X,Z) = Z^m + b_{m-1}(X)Z^{m-1} + \cdots + b_1(X)Z + b_0(X)$, $m > 1$, and $\deg(f) > 1$. We also may write $\mathcal{B} = \mathbb{K}[x,y,z]$ where x, y and z are the images of X, Y and Z under the canonical epimorphism $\mathbb{K}[X,Y,Z] \to \mathcal{B}$. Note that we have $f(x)y - \varphi(x,z) = 0$.

We show that the ML-invariant of $\mathcal{B}$ is $\mathbb{K}[x]$ (cf. Theorem 7) and, by using this result, we describe all locally nilpotent derivations of $\mathcal{B}$ (cf. Corollary 8) and its Derksen invariant (cf. Theorem 10). Further we determine a set of generators for the group of $\mathbb{K}$-automorphisms of $\mathcal{B}$ (cf. Theorem 15), when $\varphi(X,Z) \in \mathbb{K}[Z]$ and $f(X)$ has at least one nonzero root which is the case not covered by any of the previously mentioned papers. For all these results we are specially motivated by [3,11].

The material is organized as follows: Section 1 gathers the basic definitions, notations, and results used in this paper. In Section 2 we discuss several properties of