Journal of Algebra 469 (2017) 288–301

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Invariance of the restricted p-power map on integrable derivations under stable equivalences

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A R T I C L E I N F O

Article history:
Received 15 September 2015
Available online 9 August 2016
Communicated by Michel Broué

Keywords:
Hochschild cohomology
Integrable derivations
Restricted p-power map
Stable equivalences
Representation theory

A B S T R A C T

We show that the p-power maps in the first Hochschild cohomology space of finite-dimensional selfinjective algebras over a field of prime characteristic p commute with stable equivalences of Morita type on the subgroup of classes represented by integrable derivations. We show, by giving an example, that the p-power maps do not necessarily commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras.

1. Introduction

Let k be a field of prime characteristic p. For symmetric k-algebras, it is shown in [2] that the Gerstenhaber bracket in Hochschild cohomology commutes with the transfer maps introduced in [4]. Zimmermann proved in [6] that the p-power map on (the positive part of) Hochschild cohomology commutes with derived equivalences. We show in this paper that the p-power map, restricted to the classes of integrable derivations, commutes with stable equivalences of Morita type between finite-dimensional selfinjective algebras. We also show, by giving an example, that p-power maps need not commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras. To state our main result, we use the following notation: let A be a finite-dimensional selfinjective k-algebra.

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http://dx.doi.org/10.1016/j.jalgebra.2016.08.002
0021-8693/

For r a positive integer, we denote by $\mathrm{Aut}_r(A[[t]])$ the subgroup of $k[[t]]$-algebra automorphism of $A[[t]]$ which induces the identity on $A[[t]]/t^r A[[t]]$. If $\alpha \in \mathrm{Aut}_r(A[[t]])$, then there is a unique $k[[t]]$-linear map μ on $A[[t]]$ such that $\alpha(a) = a + t^r \mu(a)$ for all $a \in A[[t]]$.

An easy verification (see Proposition 2.7) shows that the map $\bar{\mu}$ induced by μ on the quotient $A[[t]]/tA[[t]] \cong A$ is a derivation; any such derivation is called *r-integrable*. We denote by $\mathrm{HH}^1_r(A)$ the image in $\mathrm{HH}^1(A)$ of all r-integrable derivations. Let A, B be finite-dimensional selfinjective k-algebras, M be an A–B-bimodule and N a B–A-bimodule. Following Broué [1], we say that M and N induce a stable equivalence of Morita type between A and B if M, N are finitely generated projective as left and right modules with the property that $M \otimes_B N \cong A \oplus X$ for some projective A–A-bimodule X and $N \otimes_A M \cong B \oplus Y$ for some projective B–B-bimodule Y. If A, B are symmetric then N can by replaced by $M^\vee = \mathrm{Hom}_k(M, k)$.

Theorem 1.1. *Let A, B be finite-dimensional selfinjective k-algebras with separable semisimple quotients, and let M, N be an A–B-bimodule, B–A-bimodule, respectively, inducing a stable equivalence of Morita type between A and B. For any positive integer r, the p-power map sends $\mathrm{HH}^1_r(A)$ to $\mathrm{HH}^1_{rp}(A)$, and we have a commutative diagram of maps*

$$\begin{array}{ccc} \mathrm{HH}^1_r(A) & \xrightarrow{\cong} & \mathrm{HH}^1_r(B) \\ \downarrow{[p]} & & \downarrow{[p]} \\ \mathrm{HH}^1_{rp}(A) & \xrightarrow{\cong} & \mathrm{HH}^1_{rp}(B) \end{array}$$

where the horizontal isomorphisms are induced by the functor $N \otimes_A - \otimes_A M$, and where the vertical maps are the p-power maps.

In Section 2 we prove the main results concerning r-integrable derivation that allow us to prove in Section 3 the Theorem 1.1. In the last section we provide an example of when the p-power map does not commute with a transfer map between the Hochschild cohomology of two symmetric algebras. For the next two sections all the tensor products are over k unless otherwise specified.

2. Integrable derivations of degree r

Let A be a finite-dimensional algebra over k. For any integer $n \geq 0$ and any $A \otimes_k A^{op}$-module M the Hochschild cohomology of degree n of A with coefficients in M is denoted by $\mathrm{HH}^n(A; M)$ in particular $\mathrm{HH}^n(A) = \mathrm{HH}^n(A; A)$. It is well known that $\mathrm{HH}^0(A) = Z(A)$ and $\mathrm{HH}^1(A)$ is the space of derivations modulo inner derivations. The direct sum $\bigoplus_{n \geq 0} \mathrm{HH}^n(A)$ is a Gerstenhaber algebra, in particular $\mathrm{HH}^1(A)$ is a Lie algebra. In addition, if k has positive characteristic p, then there is a map $[p] : \mathrm{HH}^1(A) \to$