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Admissible positive systems of affine Kac–Moody Lie algebras: The twisted cases

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A B S T R A C T

We introduce admissible positive systems and Hermitian real forms of affine twisted Kac–Moody Lie algebras, and show that a real form has admissible positive system if and only if it is Hermitian. We use the Vogan diagrams to classify the Hermitian real forms, and show that their symmetric spaces carry complex structures. The affine non-twisted Kac–Moody Lie algebras have been treated in an earlier work, and this article deals with the twisted cases.

1. Introduction

An important class of real semisimple Lie algebras $\mathfrak{s}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ is the Hermitian type, where the Riemannian symmetric space $\mathfrak{s}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}}$ has invariant Hermitian structure (see (2.4)). They play important roles in Harish-Chandra's study of representation theory, where he introduced *admissible positive systems* [6]. These are positive systems such

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that the adjoint $\mathfrak{k}$-representation on $\mathfrak{p}$ stabilizes $\mathfrak{p}^{\pm}$, and furthermore $[\mathfrak{p}^{\pm}, \mathfrak{p}^{\pm}] = 0$. This notion was generalized to the Hermitian real forms of affine non-twisted Kac–Moody Lie algebras [4]. The uniform treatment in [4] does not cover the twisted cases due to the peculiarities of their root systems. This article handles the specific features of each twisted case (for example the root system of $A^{(2)}_{\text{even}}$ has its own pattern), thereby completing the discussion of admissible positive systems of affine Kac–Moody Lie algebras initiated in [4].

Let $\mathfrak{g}$ be a complex affine Kac–Moody Lie algebra. We say that $\mathfrak{g}$ is *non-twisted*, if $\mathfrak{g} = X^{(1)}$, where X is a finite dimensional simple Lie algebra, and we say that $\mathfrak{g}$ is *twisted*, if it is one of $A^{(2)}_n$, $D^{(2)}_n$, $E^{(2)}_6$, $D^{(3)}_4$ (see [8] for more details). Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ with Cartan involution θ, and let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ be its Cartan decomposition. We drop the subscript $\mathbb{R}$ for complexification, so for example $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{h}_{\mathbb{R}}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, with root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_{\alpha}$. Let $\mathring{\mathfrak{g}} \subset \mathfrak{g}$ be the underlying finite dimensional simple Lie algebra, and let $\mathring{\Delta} \subset \Delta$ be its root system. Let δ be a generator of the imaginary roots. Every root in Δ has the form $\alpha + n\delta$, $n\delta$ or $\frac{1}{2}(\alpha + (2n-1)\delta)$, where $\alpha \in \mathring{\Delta}$ and $n \in \mathbb{Z}$ (but not all such expressions belong to Δ) [8, Prop. 6.3].

Suppose that θ acts as the identity map on $\mathfrak{h}$. Then θ acts as 1 or -1 on each $\mathfrak{g}_{\alpha}$, even if $\dim \mathfrak{g}_{\alpha} > 1$ (see Proposition 2.2). We let $\theta_{\alpha} \in \{\pm 1\}$ denote its eigenvalue. Also, θ is an extension of an involution $\mathring{\theta}$ on $\mathring{\mathfrak{g}}$. We give the definition for $\mathring{\theta}$ to be Hermitian in (2.4).

The following definition appears as [4, Def. 1.1] for non-twisted $\mathfrak{g}$, but it serves the twisted cases as well.

Definition 1.1.

(a) An *admissible* positive system for θ (or $\mathfrak{g}_{\mathbb{R}}$) is a positive system Δ^+ of a θ-stable Cartan subalgebra such that

$$\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^- \ , \ [\mathfrak{k}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm} \ , \ [\mathfrak{p}^{\pm}, \mathfrak{p}^{\pm}] = 0, \tag{1.1}$$

where $\mathfrak{p}^{\pm} = \mathfrak{p} \cap \sum_{\Delta^{\pm}} \mathfrak{g}_{\alpha}$.

(b) A real form of $\mathfrak{g}$ is said to be *Hermitian* if $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$ such that $\theta|_{\mathfrak{h}} = 1$, $\theta_{\delta} = 1$, and $\mathring{\theta}$ is Hermitian.

Let $\mathfrak{n} = \sum_{\Delta^+} \mathfrak{g}_{\alpha}$, and let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a Borel subalgebra. We have proved the following theorem for the non-twisted $\mathfrak{g}$ [4, Thms. 1.2, 1.5]. In this article, we prove it for the twisted $\mathfrak{g}$.

Theorem 1.2. *There exists an admissible positive system for $\mathfrak{g}_{\mathbb{R}}$ if and only if $\mathfrak{g}_{\mathbb{R}}$ is Hermitian. If these equivalent conditions are satisfied, we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}} + \mathfrak{n}$, and there exists a parabolic subalgebra $\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{g}$ which provides a complex structure on $\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}}$ by $\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}} \cong \mathfrak{g}/\mathfrak{q}$.*