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Flat affine or projective geometries on Lie groups [☆]

A. Medina [a,b], O. Saldarriaga [c,*], H. Giraldo [c]

[a] Université Montpellier, Institute A. Grothendieck, UMR 5149 du CNRS, France
[b] Universidad de Antioquia, Colombia
[c] Instituto de Matemáticas, Universidad de Antioquia, Colombia

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This paper deals essentially with affine or projective transformations of Lie groups endowed with a flat left invariant affine or projective structure. These groups are called flat affine or flat projective Lie groups. We give necessary and sufficient conditions for the existence of flat left invariant projective structures on Lie groups. We also determine Lie groups admitting flat bi-invariant affine or projective structures. These groups could play an essential role in the study of homogeneous spaces $M = G/H$ having a flat affine or flat projective structures invariant under the natural action of G on M. A. Medina asked several years ago if the group of affine transformations of a flat affine Lie group is a flat projective Lie group. In this work we provide a partial positive answer to this question.

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* Corresponding author.
E-mail addresses: alberto.medina@univ-montp2.fr (A. Medina), omar.saldarriaga@udea.edu.co (O. Saldarriaga), hernan.giraldo@udea.edu.co (H. Giraldo).

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0. Introduction

One of the aims of this paper is to give a positive answer to a question raised by the first author several years ago. More precisely we prove, in several cases, that the group $\mathit{Aff}(G,\nabla^+)$ of affine transformations of a Lie group G endowed with a flat left invariant affine structure ∇^+, admits a flat left invariant projective (some times affine) structure.

Recall that a locally affine manifold is a smooth manifold M endowed with a flat and torsion free linear connection ∇. This means that the corresponding affine connection ∇ is flat. In this case the pair (M,∇) is called a flat affine manifold. We will suppose manifolds to be real connected unless otherwise stated.

The set of diffeomorphisms $\mathit{Aff}(M,\nabla)$ of M preserving ∇ endowed with the open compact topology and composition is a Lie group (see [21] page 229). That $F \in \mathit{Aff}(M,\nabla)$ means that F verifies the system of partial differential equations $F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y$, where F_* is the differential of F and X and Y are smooth vector fields on M. In particular F preserves geodesics, but in general the group of geodesic preserving diffeomorphisms of M is larger than $\mathit{Aff}(M,\nabla)$.

The aim of flat affine geometry is the study of flat affine manifolds. The local model of real flat affine geometry is the n-dimensional real affine space $\mathbb{A}^n$ endowed with the usual affine structure ∇^0. This one is given in standard notation by

$$\nabla^0_X Y = \sum_{j=1}^{n} X(g_j)\partial_j, \qquad \text{for} \qquad Y = \sum_{j=1}^{n} g_j \partial_j$$

with X and Y smooth vector fields in $\mathbb{A}^n$.

From now on we identify $\mathbb{A}^n$ with $\mathbb{R}^n$. We will often see $\mathbb{R}^n$ as the affine subspace $\{(x,1) \mid x \in \mathbb{R}^n\}$ of $\mathbb{R}^{n+1}$. Hence the classical affine group $\mathit{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes_{Id_{GL(\mathbb{R}^n)}} GL(\mathbb{R}^n)$ of affine transformations of $(\mathbb{R}^n,\nabla^0)$ will be identified with a closed subgroup of $GL(\mathbb{R}^n \oplus \mathbb{R})$.

Notice also that every invariant pseudo metric in $\mathbb{R}^n$ determines the same geodesics as ∇^0.

Definition 1. A flat affine (respectively a flat projective) Lie group is a Lie group endowed with a flat left invariant affine structure (respectively flat left invariant projective structure). The first one will be abbreviated as FLIAS. The corresponding infinitesimal object is called an affine Lie algebra (respectively a projective Lie algebra).

For the local model of flat projective geometry see Section 1.

An important and difficult open problem is to determine whether a manifold (respectively a Lie group) admits a flat (respectively left invariant) affine or projective structure. Obviously a manifold $H\backslash G$, where H is a discrete co-compact subgroup of a flat affine Lie group G, inherits a flat affine structure. Let G be a Lie group of Lie algebra $\mathfrak{g} := T_\epsilon(G)$ with ϵ the unit of G. If $x \in T_\epsilon(G)$, the left invariant vector field