Journal of Algebra 414 (2014) 271–287

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Irreducible Virasoro modules from irreducible Weyl modules

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A R T I C L E I N F O

Article history:
Received 19 December 2013
Available online 16 June 2014
Communicated by Volodymyr Mazorchuk

MSC:
17B10
17B20
17B65
17B66
17B68

Keywords:
The Virasoro algebra
Non-weight irreducible module
Weyl module
Twisting technique

A B S T R A C T

We use Block's results to classify irreducible modules over the differential operator algebra $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$. From modules A over $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ and using the "twisting technique" we construct a class of modules A_b over the Virasoro algebra for any $b \in \mathbb{C}$. These new Virasoro modules are generally not weight modules. The necessary and sufficient conditions for A_b to be irreducible are obtained. Then we determine the necessary and sufficient conditions for two such irreducible Virasoro modules to be isomorphic. Many interesting examples for such irreducible Virasoro modules with different features are provided at the end of the paper. In particular the class of irreducible Virasoro modules $\Omega(\lambda, b)$ for any $\lambda \in \mathbb{C}^*$ and any $b \in \mathbb{C}$ are defined on the polynomial algebra $\mathbb{C}[x]$.

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http://dx.doi.org/10.1016/j.jalgebra.2014.04.029

1. Introduction

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, real numbers and complex numbers, respectively.

Let $\mathfrak{V}$ denote the complex *Virasoro algebra*, that is the Lie algebra with basis $\{c, d_i : i \in \mathbb{Z}\}$ and the Lie bracket defined (for $i, j \in \mathbb{Z}$) as follows:

$$[d_i, d_j] = (j-i)d_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c; \qquad [d_i, c] = 0.$$

The algebra $\mathfrak{V}$ is one of the most important Lie algebras both in mathematics and in mathematical physics, see for example [13,9] and references therein. The Virasoro algebra theory has been widely used in many physics areas and other mathematical branches, for example, quantum physics [7], conformal field theory [6], higher-dimensional WZW models [11,10], Kac–Moody algebras [12,22], vertex algebras [14], and so on.

The representation theory of the Virasoro algebra has been attracting a lot of attention from mathematicians and physicists. There are two classical families of simple Harish-Chandra $\mathfrak{V}$-modules: highest weight modules (completely described in [2]) and the so-called intermediate series modules. In [18] it is shown that these two families exhaust all simple weight Harish-Chandra modules. In [20] it is even shown that the above modules exhaust all simple weight modules admitting a nonzero finite dimensional weight space.

Naturally, the next task is to study irreducible non-weight modules and irreducible weight modules with infinite dimensional weight spaces. Irreducible weight modules with infinite dimensional weight spaces were firstly constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [28] in 1997, and the necessary and sufficient conditions for such tensor product to be simple were recently obtained in [3]. Conley and Martin gave another class of such examples with four parameters in [4] in 2001. Then very recently, new weight simple Virasoro modules were found in [16,17].

Various families of irreducible non-weight Virasoro modules were studied in [23,15, 5,27,8,24,19]. These include various versions of Whittaker modules constructed using different tricks. In particular, all the above Whittaker modules and even more were described in a uniform way in [21].

The main purpose of the present paper is to construct new irreducible (non-weight) Virasoro modules. Let us first introduce the algebras we will use.

Let $\mathbb{C}[t]$ be the (associative) polynomial algebra. By ∂ we denote the operator $t\frac{\mathrm{d}}{\mathrm{d}t}$ on $\mathbb{C}[t]$. We see that $\partial t^i = t^i(\partial + i)$. Then we have the associative algebra $\mathcal{A} = \mathbb{C}[t, \partial]$ which is a proper subalgebra of the rank 1 Weyl algebra $\mathbb{C}[t, \frac{\mathrm{d}}{\mathrm{d}t}]$. Note that $\mathcal{A}$ is the universal enveloping algebra of the 2-dimensional solvable Lie algebra $\mathfrak{a}_1 = \mathbb{C}d_0 \oplus \mathbb{C}d_1$ subject to $[d_0, d_1] = d_1$. See [1,21]. Let $\mathcal{K} = \mathbb{C}[t, t^{-1}, \partial]$ be the Laurent polynomial differential operator algebra.