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New convolutions for the number of divisors

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We introduce new convolutions for the number of divisors function. We also provide combinatorial interpretations for some of the convolutions. In addition, we prove arithmetic properties for several restricted partitions functions used in the convolutions.

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1. Introduction

The divisors of numbers have been studied from the point of view of partitions of integers for a long time [5]. It is well-know that Euler's partition function $p(n)$ [1] and the sum of divisors function

$$\sigma(n) = \sum_{d|n} d$$

satisfy common recursive relations with only $p(0)$ different from $\sigma(0)$:

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} p(n - G_k) = \delta_{0,n}, \quad \text{with} \quad p(0) = 1,$$

and

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} \sigma(n - G_k) = 0, \quad \text{with} \quad \sigma(0) \text{ replaced by } n,$$

where $\delta_{i,j}$ is the Kronecker delta and

$$\{G_k\}_{k \geqslant 0} = \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \dots\}$$

is the sequence of generalized pentagonal numbers, i.e.,

$$G_k = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil.$$

We use the established convention $p(k) = \sigma(k) = 0$ if $k < 0$.

Moreover, there are two relations that combine the functions $p(n)$ and $\sigma(n)$,

$$np(n) = \sum_{k=0}^{n} \sigma(k) p(n-k) \tag{1}$$

and

$$\sigma(n) = \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil - 1} G_k p(n - G_k). \tag{2}$$

Note that in (1) and (2), $\sigma(0) = 0$. More details about these classical connections between partitions and divisors can be found in [8] and the references therein.

Recently, motivated by these results, Merca [6] proved that the number of divisors function

$$\tau(n) = \sum_{d|n} 1$$