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# The absolute trace of totally positive reciprocal algebraic integers

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ABSTRACT

In this paper, we compute the lower bound of the absolute trace of totally positive reciprocal algebraic integers with the use of an explicit auxiliary function and prove that all but finitely many totally positive reciprocal algebraic integers $\alpha$ have $\mathrm{tr}(\alpha)/d > 1.8945909\cdots$.



## 1. Introduction

Let $\alpha$ be a totally positive algebraic integer of degree $d$, with minimal polynomial

$$P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0,$$

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$\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_d$ its conjugates. Let $\mathrm{tr}(\alpha) = \sum_{j=1}^{d} \alpha_j$, then $\mathrm{tr}(\alpha)$ is the trace of $\alpha$, $\mathrm{tr}(\alpha)/d$ is called the absolute trace of $\alpha$. On the absolute trace of the totally positive algebraic integer, there is a famous "Schur–Siegel–Smyth trace problem [5]", that is

**Fix $\rho < 2$. Then show that all but finitely many totally positive algebraic integers $\alpha$ have $\mathrm{tr}(\alpha)/d > \rho$.**

In 1918 Schur [13] showed that $\rho = \sqrt{e}$ with one exceptional polynomial $x^2 - 3x + 1$; in 1945 Siegel [14] showed that $\rho = 1.733\cdots$ with another one exceptional polynomial $x^3 - 5x^2 + 6x - 1$; in 1984 Smyth [15] proved that $\rho = 1.7719\cdots$ with two new exceptional polynomials $x^4 - 7x^3 + 13x^2 - 7x + 1$ and $x^4 - 7x^3 + 14x^2 - 8x + 1$. After that many people study this problem, we have $\rho = 1.7735\cdots$ in 1997 by Flammang et al. [8]; $\rho = 1.7783\cdots$ in 2004 by McKee and Smyth [12]; $\rho = 1.78002\cdots$ in 2006 by Aguirre et al. [1]; $\rho = 1.7836\cdots$ in 2007 by Aguirre and Peral [2]; $\rho = 1.7841\cdots$ in 2008 by Aguirre and Peral [3]; $\rho = 1.78702\cdots$ in 2009 by Flammang [7] and $\rho = 1.7919\cdots$ in 2011 by Liang and Wu [11], but no other exceptional polynomial was found. For this problem, we can use the following explicit auxiliary function

$$f(x) = x - \sum_{i=1}^{n} e_i \log |Q_i(x)|,$$

where $x \in (0, \infty)$, $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{R}^n$, $e_i \geq 0$, $Q_i \in \mathbb{Z}[x]$, $1 \leq i \leq n$.

In fact, if we suppose that $m(\mathbf{e}) = \min_{x>0} f(x, \mathbf{e})$. Then we have

$$\alpha_j - \sum_{i=1}^{n} e_i \log |Q_i(\alpha_j)| \geq m(\mathbf{e}), \text{ for } 1 \leq j \leq d,$$

and therefore,

$$\sum_{j=1}^{d} \alpha_j - \sum_{i=1}^{n} e_i \log \left| \prod_{j=1}^{d} Q_i(\alpha_j) \right| \geq d \cdot m(\mathbf{e}),$$

$$\mathrm{tr}(\alpha) - \sum_{i=1}^{n} e_i \log |\mathrm{Res}(P, Q_i)| \geq d \cdot m(\mathbf{e}).$$

If $P$ doesn't divide any of the $Q_i$, $\mathrm{Res}(P, Q_i)$ is a nonzero integer, thus $\mathrm{tr}(\alpha)/d \geq m(\mathbf{e})$.

On the other hand, in 1999, Serre proved [3] that any lower bound of the absolute trace of totally positive algebraic integers found by the method using the explicit auxiliary function as above cannot be larger than 1.8983021.

In this paper, we are interested in the absolute trace of totally positive **reciprocal** algebraic integer, we constructed a new type explicit auxiliary function using the properties of reciprocal algebraic integers, and we have.