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A multiquadratic field generalization of Artin's conjecture

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We prove (under the assumption of the generalized Riemann hypothesis) that a totally real multiquadratic number field K has a positive density of primes p in $\mathbb{Z}$ for which the image of $\mathcal{O}_K^\times$ in $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ has minimal index $(p-1)/2$ if and only if K contains a unit of norm -1. An explicit formula for this density is provided. We also discuss an application to ray class fields of conductor $p\mathcal{O}_K$.

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1. Introduction

In 1927, Emil Artin [1] conjectured that for any integer a not equal to ± 1 or a square, there is a positive density of primes p for which a is a primitive root in $\mathbb{F}_p^\times$. In 1967, Hooley [5] proved that this density exists and is positive if a set of generalized Riemann hypotheses (GRH) is true for particular number fields. Many authors have since adapted the analytic methods of Hooley to different settings and proved that the GRH implies a variety of interesting results. Weinberger [21] showed that if the ring of integers is a principal ideal domain, then it is a Euclidean domain for certain number fields. Cooke and Weinberger [3] proved that all 2×2 matrices in the special linear group of certain rings can be expressed as a product of nine elementary matrices. Matthews [12] generalized Hooley's argument to count the number of primes for which each element of a set of integers is a primitive root mod p. Necessary and sufficient conditions for the conjectural density to be nonzero in a more general global field setting were found by Lenstra [10]. He also provided an application of his results to the existence of a Euclidean algorithm in certain arithmetic rings. In the 1980s, Murty [15] generalized Hooley's methods to count prime ideals in any family of normal finite extension fields of a number field K and included an application to elliptic curves. His methods were generalized by Roskam [16,17] to prove analogs of Artin's conjecture in quadratic fields in the early 2000s. More recently, Lenstra, Stevenhagen, and Moree [11] interpreted the correction factors that arise in the density computations as character sums describing how the particular number fields are entangled and provided an application to Serre curves.

In [2], Cangelmi and Pappalardi generalized Artin's conjecture from an integer a to a finitely generated multiplicative subgroup of $\mathbb{Q}^\times$ and proved there is a positive density of primes p for which reducing the elements of the subgroup mod p yields the full multiplicative group of coprime residue classes mod p. Roskam [16] generalized the conjecture from $\mathbb{F}_p^\times$ to $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ for a real quadratic field K and proved there is a positive density of primes p for which the image of a fundamental unit ϵ in $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ is maximal (or equivalently, has index as small as possible). We further generalize these results and investigate when the image of the full unit group $\mathcal{O}_K^\times$ in $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ has index as small as possible for a positive density of primes p and prove a result for mutiquadratic fields. When K is a totally real multiquadratic field that is not quadratic, $\mathcal{O}_K^\times$ has rank greater than 1 and so there is not a single natural element to consider such as a fundamental unit. Thus we decided to formulate a generalization using the entire