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A note on mass equidistribution of holomorphic Siegel modular forms ☆

Sheng-Chi Liu

Department of Mathematics, Washington State University, Pullman, WA 99164-3113, USA

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ABSTRACT

Let $F_f \in S_k(Sp_{2n}(\mathbb{Z}))$ be the Ikeda lifting of a Hecke eigenform $f \in S_{2k-n}(SL_2(\mathbb{Z}))$ with the normalization $\langle F_f, F_f\rangle = 1$. Let $E(Z; s)$ denote the Klingen Eisenstein series. In this paper we verify that

$$\lim_{k\to\infty} \int\limits_{Sp_{2n}(\mathbb{Z})\backslash\mathfrak{H}_n} E(Z; \tfrac{n}{2} + it)|F_f(Z)|^2(\det Y)^k d\mu = 0$$

which is predicted by the mass equidistribution conjecture of Cogdell and Luo [CL].

1. Introduction

Let $n \geq 2$ be a positive integer. Let $\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}) : Z = {}^tZ,\ \mathrm{Im}Z > 0\}$ be the Siegel upper half-space of degree n, and let $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ be the Siegel modular group. Γ_n acts discontinuously on $\mathfrak{H}_n$ by

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E-mail address: scliu@math.wsu.edu.

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$$g\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \qquad \text{for } Z \in \mathfrak{H}_n, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

Denote by $S_k(\Gamma_n)$ the space of Siegel cusp forms of degree n and weight k. For $F, G \in S_k(\Gamma_n)$, we define the Petersson inner product

$$\langle F, G\rangle = \int\limits_{\Gamma_n\backslash\mathfrak{H}_n} F(Z)\overline{G(Z)}(\det Y)^k d\mu$$

where $d\mu = \dfrac{dXdY}{(\det Y)^{n+1}}$ is the Γ_n-invariant measure. The following mass equidistribution conjecture was formulated by Cogdell and Luo [CL].

Conjecture 1.1 *(Mass equidistribution). Let $\{F_{k,j}\}$ be an orthonormal Hecke eigenbasis for $S_k(\Gamma_n)$. For $K \subset \Gamma_n\backslash\mathfrak{H}_n$ compact, we have*

$$\lim_{k\to\infty} \int\limits_K |F_{k,j}(Z)|^2(\det Y)^k d\mu = \frac{1}{vol(\Gamma_n\backslash\mathfrak{H}_n)} \int\limits_K d\mu. \tag{1.1}$$

By Weyl's criterion, the conjecture is equivalent to

$$\lim_{k\to\infty} \int\limits_{\Gamma_n\backslash\mathfrak{H}_n} \phi(Z)|F_{k,j}(Z)|^2(\det Y)^k d\mu = \frac{1}{vol(\Gamma_n\backslash\mathfrak{H}_n)} \int\limits_{\Gamma_n\backslash\mathfrak{H}_n} \phi(Z)d\mu \tag{1.2}$$

for any ϕ in the spectrum of $L^2(\Gamma_n\backslash\mathfrak{H}_n)$. In this note, we shall verify (1.2) when ϕ equals the Klingen Eisenstein series and $F_{k,j}$ is an Ikeda lifting.

2. Main result

2.1. The Klingen Eisenstein series

Let P_n denote the subgroup of Γ_n consisting of all those matrices with $(0, \cdots, 0, 1)$ as its last row. The Klingen Eisenstein series is defined as

$$E(Z;s) := \frac{1}{2}\sum_{g\in P_n\backslash\Gamma_n} \left(\frac{\det \operatorname{Im} g\langle Z\rangle}{\det \operatorname{Im}(g\langle Z\rangle)_1}\right)^s$$

for $Z \in \mathfrak{H}_n$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > n$. Here the subscript 1 denotes the upper left $(n-1)\times(n-1)$ corner of an $n\times n$ matrix. The Eisenstein series satisfies the functional equation

$$E^*(Z;s) := \pi^{-s}\Gamma(s)\zeta(2s)E(Z;s) = E^*(Z;n-s).$$