Journal of Number Theory 170 (2017) 250–262

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

More infinite families of congruences modulo 5 for broken 2-diamond partitions

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ARTICLE INFO

Article history:
Received 19 January 2016
Accepted 14 April 2016
Available online 2 June 2016
Communicated by David Goss

MSC:
11P83
05A17

Keywords:
Broken k-diamond partition
Congruence
Theta function

ABSTRACT

The notion of broken k-diamond partitions was introduced by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Recently, Chan, and Paule and Radu proved some congruences modulo 5 for $\Delta_2(n)$. In this paper, we prove several new infinite families of congruences modulo 5 for $\Delta_2(n)$ by using an identity due to Newman. Our results generalize the congruences proved by Paule and Radu.

1. Introduction

The aim of this paper is to prove several new infinite families of congruences modulo 5 for broken 2-diamond partitions, which generalize some congruence results due to Paule and Radu [11].

Let us begin with some notation and terminology on q-series and partitions. We use the standard notation

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http://dx.doi.org/10.1016/j.jnt.2016.04.023

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and often write

$$(a_1,a_2,\dots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.1}$$

where $|ab|<1$. The Jacobi triple product identity can be restated as

$$f(a,b)=(-a,-b,ab;ab)_\infty. \tag{1.2}$$

One special case of (1.1) is defined by

$$\psi(q)=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}. \tag{1.3}$$

By (1.1), (1.2) and (1.3),

$$\psi(q)=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{1.4}$$

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [2] to the construction of a new class of directed graphs called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Andrews and Paule [2] established the following generating function of $\Delta_k(n)$:

$$\sum_{n=0}^{\infty}\Delta_k(n)q^n=\frac{(q^2;q^2)_\infty(q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3(q^{4k+2};q^{4k+2})_\infty}. \tag{1.5}$$

Employing generating function manipulations, Andrews and Paule [2] proved that for all integers $n\geq 0$,

$$\Delta_1(2n+1)\equiv 0 \pmod 3.$$

Since then, a number of congruences satisfied by $\Delta_k(n)$ for small values of k have been proved. See, for example, Chan [4], Chen, Fan and Yu [5], Hirschhorn and Sellers [6], Lin [7], Lin and Wang [8], Paule and Radu [11], Radu and Sellers [12–14], Wang and Yao [18], Xia [15,16] and Yao [17].