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Families of polynomials and their specializations

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For a polynomial in several variables depending on some parameters, we discuss some results to the effect that for almost all values of the parameters the polynomial is irreducible. In particular we recast in this perspective some results of Grothendieck and of Gao.

0. Introduction

This paper is devoted to irreducibility questions for families of polynomials in several indeterminates $x_1, \ldots, x_\ell$ parametrized by further indeterminates $t_1, \ldots, t_s$. We assume

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that $\ell \geqslant 2$ and the base field k is algebraically closed; the more arithmetic case $\ell = 1$ depends on the base field and involves different tools and techniques.

Set $\underline{t} = \{t_1, \ldots, t_s\}$, $\underline{x} = \{x_1, \ldots, x_\ell\}$ and consider a polynomial $F \in k[\underline{t}, \underline{x}]$, irreducible in $\overline{k(\underline{t})}[\underline{x}]$ (where $\overline{k(\underline{t})}$ is the algebraic closure of $k(\underline{t})$); F is said to be *generically irreducible*. The core question is about the irreducibility of the polynomials obtained by substituting elements $t_1^*, \ldots, t_s^* \in k$ for the corresponding parameters $t_1, \ldots, t_s$ – the *specializations of* F.

More specifically we wish to investigate the following problem, as explicitly as possible:

– when the generic irreducibility property is satisfied, show some boundedness results on the following set, which we call the *spectrum* of F:

$$\mathrm{sp}(F) = \{\underline{t}^* = (t_1^*, \ldots, t_s^*) \in k^s \mid F(\underline{t}^*, \underline{x}) \text{ is reducible in } k[\underline{x}]\},$$

and some density results for its complement,

– find some criteria for the generic irreducibility property to be satisfied and deduce some new specific examples.

A first approach rests on classical results of Noether and Bertini and a second one involves more combinatorial tools like the Newton polygon and the associated Minkowski theorem. We contribute to these approaches by implementing some ideas and results coming from connected areas, notably of Grothendieck (Arithmetic Geometry) and Gao (Polyhedral Combinatorics). This leads to new answers to the problem together with an improved and unified presentation of results from our previous papers [BDN09a,BDN09b] and other related papers. Those were concerned with special cases of the general situation considered here. In particular polynomials $f(x, y) - t$ and variants of those have been much studied and the word "spectrum" refers to the classical terminology used in this special case.

§1 briefly reviews the classical background and introduces our contribution, which is then detailed in §2 and §3.

1. The classical approaches and our contribution

1.1. The arithmetico-geometric approach

Fix $F \in k[\underline{t}, \underline{x}]$ and assume as above that it is generically irreducible.

1.1.1. Noether

Denote by $\mathcal{U}_F$ the open Zariski subset of all $\underline{t}^* \in k^s$ such that $\deg(F(\underline{t}^*, \underline{x})) = \deg_{\underline{x}} F(\underline{t}, \underline{x})$. The spectrum $\mathrm{sp}(F)$ is a proper Zariski closed subset of $\mathcal{U}_F$: there exist non-zero polynomials $h_1, \ldots, h_\nu \in k[\underline{t}]$ such that

$$(1) \quad \mathrm{sp}(F) \cap \mathcal{U}_F = \mathcal{Z}(h_1, \ldots, h_\nu) \cap \mathcal{U}_F$$

where $\mathcal{Z}(h_1, \ldots, h_\nu)$ denotes the zero set of $h_1, \ldots, h_\nu$. In other words, for $\underline{t}^* \in k^s$ such that $\deg(F(\underline{t}^*, \underline{x})) = \deg_{\underline{x}} F(\underline{t}, \underline{x})$,