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New properties of the Lerch's transcendent

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A new representation of the Lerch's transcendent $\Phi(z,s,a)$, valid for positive integer $s=n=1,2,\ldots$ and for z and a belonging to certain regions of the complex plane, is presented. It allows to write an equation relating $\Phi(z,n,a)$ and $\Phi(1/z,n,1-a)$, which in turn provides an expansion of $\Phi(z,n,a)$ as a power series of $1/z$, convergent for $|z|>1$.

1. Introduction

The Lerch's transcendent $\Phi(z,s,a)$, also known as Hurwitz–Lerch zeta function, is defined by its series representation [2, Sec. 1.11, Eq. (1)], [9, Eq. 25.14.1]

$$\Phi(z,s,a)=\sum_{m=0}^{\infty}\frac{z^m}{(a+m)^s}, \tag{1}$$

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provided

$$a \neq 0, -1, -2, \ldots; \qquad |z| < 1; \qquad |z| = 1, \quad \Re s > 1. \tag{2}$$

The restriction on the values of a guarantees that all terms of the series in the right-hand side are finite. Obviously, the series is convergent if $|z| < 1$, independently of the value of s, or if $|z| = 1$ and $\Re s > 1$. For other values of its arguments, $\Phi(z, s, a)$ is defined by analytic continuation. This is achieved by means of integral representations, the most common of them being [9, Eq. 25.14.5]

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, e^{-at}}{1 - z\, e^{-t}}\, dt, \tag{3}$$

whenever

$$\Re s > 0, \qquad \Re a > 0, \qquad z \in \mathbb{C} \setminus [1, \infty). \tag{4}$$

The conditions imposed on a and z ensure the regularity of the integrand in the right-hand side of (3). The restriction on s allows to prove the equivalence of the representations (1) and (3) in their common region of validity (see, for instance, [6, Lemmas 2.1 and 2.2]). A thorough discussion of the analytic continuation of Φ, as a multivalued function of three complex variables, and of its "singular strata" can be found in a recent paper by Lagarias and Li [7], where the monodromy functions describing the multivaluedness are computed. (Be aware that, in the notation used in Ref. [7], the two first arguments of Φ are transposed, as compared with the notation used in Refs. [2] and [9].)

In the course of a research on the use of dispersion relations in the study of elementary particles [4,5], we have encountered what we believe to be a new representation of the Lerch's transcendent $\Phi(z, s, a)$ for positive integer values of the second argument, $s = n = 1, 2, \ldots$. This is our first result, presented as Theorem 1 in Sec. 2. Such a representation allows to unveil, as a second result reported in Sec. 3 as Theorem 2, a property of Φ not noticed before. This property, in turn, provides our third result, expressed in Corollary 1 of Sec. 4, consisting of an expansion of $\Phi(z, n, a)$ in powers of $1/z$, convergent for $|z| > 1$. The proofs of the three results, followed by pertinent remarks, are presented in Secs. 2, 3 and 4, respectively. Some comments are added in Sec. 5.

2. A new representation

Let $\mathbf{D}$ denote the open unit disc in the complex plane, cut along the negative real semiaxis, that is,

$$z \in \mathbf{D} \qquad \Rightarrow \qquad z \in \mathbb{C}, \quad 0 < |z| < 1, \quad -\pi < \arg(z) < \pi. \tag{5}$$