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# On the functional equation of the Siegel series

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ABSTRACT

It is well-known that the Fourier coefficients of Siegel–Eisenstein series can be expressed in terms of the Siegel series. The functional equation of the Siegel series of a quadratic form over $\mathbb{Q}_p$ was first proved by Katsurada. In this paper, we prove the functional equation of the Siegel series over a non-archimedean local field of characteristic 0 by using the representation theoretic argument by Kudla and Sweet.


## 0. Introduction

The theory of Siegel series was initiated by Siegel [14] to investigate the Fourier coefficients of the Siegel Eisenstein series. Since then, many authors treated Siegel series. Katsurada [4] gave an explicit formula for the Siegel series over $\mathbb{Q}_p$. To obtain the explicit formula, Katsurada proved a functional equation of the Siegel series, which is now called the Katsurada functional equation. The purpose of this paper is to generalize Katsurada functional equation over an arbitrary local field of characteristic not 2.

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There are several proofs of the Katsurada functional equation over $\mathbb{Q}_p$. Böcherer and Kohnen [1] used the global functional equation of the Siegel Eisenstein series. The proof of Sato and Hironaka [10] used the theory of spherical functions. In fact, Karel [3] has shown that there exists a functional equation by using the representation theory, but he did not calculate a precise form of the functional equation. The precise form of the functional equation can be calculated by using the result of Sweet [15] on the "gamma matrix" of a prehomogeneous vector space, in principle.

In this paper, we first reformulated the result of Sweet [15] suitable for our purpose. Let $\mathrm{Sym}_n(F)$ be the space of symmetric matrices over a non-archimedean local field $F$ of characteristic not 2. We will calculate a precise form of the local functional equation of the prehomogeneous vector space $\mathrm{Sym}_n(F)$. Our method of the calculation is basically the same as that of Sato [9].

We now explain the content of this paper. In section 1, we give a preliminary result on the Weil constants and Tate's local factors. In section 2, we give a local functional equation (Theorem 2.1 and Theorem 2.2) for the prehomogeneous vector space $\mathrm{Sym}_n(F)$. In these theorems, we consider the zeta integrals with respect to a character $\omega$ of $F^\times$. For $\omega = 1$, our functional equation reduces to the result of Sweet [15]. In section 3, we explain the relation of the functional equation of the prehomogeneous vector space $\mathrm{Sym}_n(F)$ and that of the degenerate Whittaker functional of the degenerate principal series of $\mathrm{Sp}_n(F)$. Note that this relation was established for unitary groups in Kudla and Sweet [5]. Combining these results, we prove the functional equation of the Siegel series in section 4.

## 1. Weil constants and Tate's local factors

Let $F$ be a non-archimedean local field whose characteristic is not 2. The maximal order of $F$ and its maximal ideal is denoted by $\mathfrak{o}$ and $\mathfrak{p}$, respectively. The number of elements of the residue field $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$ is denoted by $q$. For $x \in F^\times$, we have $q^{-\mathrm{ord} x} = |x|$. The Haar measure $dx$ on $F$ is normalized so that $\int_{\mathfrak{o}} dx = 1$. The Hilbert symbol of $F$ of degree 2 is denoted by $\langle\ ,\ \rangle$. We put $F^{\times 2} = \{x^2 \mid x \in F^\times\}$. Similarly, put $\mathfrak{o}^{\times 2} = \{x^2 \mid x \in \mathfrak{o}^\times\}$. It is well-known that $[F^\times : F^{\times 2}] = 4\,|2|^{-1}$ and $[\mathfrak{o}^\times : \mathfrak{o}^{\times 2}] = 2\,|2|^{-1}$. For $\theta \in F^\times/F^{\times 2}$, we put $\chi_\theta(x) = \langle \theta, x\rangle$.

We fix a non-trivial additive character $\psi$ of $F$. Let $c_\psi$ be the order of $\psi$, i.e., $c_\psi$ is the maximal integer $c$ such that $\psi$ is trivial on $\mathfrak{p}^{-c}$. We fix an element $\boldsymbol{\delta} \in F^\times$ such that $\mathrm{ord}(\boldsymbol{\delta}) = c_\psi$.

For each Schwartz function $\phi \in \mathcal{S}(F)$, the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = |\boldsymbol{\delta}|^{1/2} \int_F \phi(y)\psi(xy)\, dy.$$

Note that the Haar measure $|\boldsymbol{\delta}|^{1/2} dy$ is the self-dual Haar measure for the Fourier transform $\phi \mapsto \hat{\phi}$. For each $a \in F^\times$, there exists a constant $\alpha_\psi(a)$, called the Weil constant, which satisfies