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On algebraic independence of a class of infinite products

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ABSTRACT

Algebraic independence of values of certain infinite products is proved, where the transcendence of such numbers was already established by Tachiya. As applications explicit examples of algebraically independent numbers are also given.

1. Introduction and the main results

Let K be an algebraic number field and $\mathcal{O}_K$ the ring of integers of K. For $\alpha \in K$ we shall denote the size of α by $\|\alpha\| = \max(|\overline{\alpha}|, \operatorname{den}(\alpha))$, where $|\overline{\alpha}|$ is the maximum of the absolute values of its conjugates and $\operatorname{den}(\alpha)$ is the least positive integer such that $\operatorname{den}(\alpha)\alpha \in \mathcal{O}_K$. In the present paper we are interested in infinite products of the form

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$$\Phi(z) = \prod_{k=0}^{\infty} \frac{E_k(z^{r^k})}{F_k(z^{r^k})}, \tag{1}$$

where $r \geq 2$ is an integer and

$$E_k(z) = 1 + a_{k,1}z + \cdots + a_{k,L}z^L,\ F_k(z) = 1 + b_{k,1}z + \cdots + b_{k,L}z^L \in K[z] \tag{2}$$

with an integer $L \geq 1$.

In [7] Tachiya proved that under some conditions $\Phi(\alpha)$ with algebraic α, $0 < |\alpha| < 1$, is algebraic if and only if $\Phi(z) \in K(z)$. For this proof he applied the method developed in [3], which is based on inductive argument introduced in [2] and a variant of Mahler's method created by Loxton and van der Poorten in [5]. Here it is essential that a sequence of functions

$$\Psi_n(z) = \prod_{k=0}^{\infty} \frac{E_{n+k}(z^{r^k})}{F_{n+k}(z^{r^k})}, \quad n \geq 0,$$

(note that $\Psi_0(z) = \Phi(z)$) satisfies a chain of Mahler type functional equations

$$\Psi_{n+1}(z^r) = \frac{F_n(z)}{E_n(z)}\Psi_n(z), \quad n \geq 0. \tag{3}$$

The results of [7] were further developed in [1], in particular a quantitative refinement of Tachiya's result was obtained in [1, Theorem 4] if the irrationality measure of the function $\Phi(z)$ is finite. Recall that the irrationality measure $\mu(f)$ of $f(z) \in K[[z]]$ is defined to be the infimum of μ such that

$$\operatorname{ord}(A(z)f(z) - B(z)) \leq \mu M,$$

holds for all nonzero $(A(z), B(z)) \in K[z]^2$ satisfying $\max(\deg A, \deg B) \leq M$ provided that $M \geq M_0$ with some sufficiently large M_0 depending only on $f(z)$, where for $g(z) \in K[[z]]$ we denote by $\operatorname{ord} g(z)$ the zero order of $g(z)$ at $z = 0$. If there does not exist such a μ, we define $\mu(f) := \infty$. Note that the condition $\mu(\Phi) < \infty$ can be verified in some cases by using [1, Lemma 9].

Remark 1. Under the above notations, if $f(z) \in K(z)$, then $\mu(f) = \infty$, but $\operatorname{ord}(A(z)f(z) - B(z)) \leq 2M$ when $A(z)f(z) - B(z) \neq 0$ and M is at least the maximum of the degrees of the numerator and the denominator of $f(z)$.

Our main aim here is to study the algebraic independence of values of several products of type (1). Let

$$\Phi_j(z) = \prod_{k=0}^{\infty} \frac{E_{j,k}(z^{r^k})}{F_{j,k}(z^{r^k})}, \quad j = 1, \ldots, m, \tag{4}$$