Journal of Number Theory 172 (2017) 160–177

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

New primitive covering numbers and their properties

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ARTICLE INFO

Article history:
Received 3 June 2016
Received in revised form 30 August 2016
Accepted 30 August 2016
Available online 8 October 2016
Communicated by D. Goss

MSC:
primary 11B25, 11A07, 05A99

Keywords:
Covering system
Covering number
Primitive covering number
Congruence

ABSTRACT

A *covering number* is a positive integer L such that a covering system of the integers can be constructed with distinct moduli that are divisors $d > 1$ of L. If no proper divisor of L is a covering number, then L is called *primitive*. In 2007, Zhi-Wei Sun gave sufficient conditions for the existence of infinitely many covering numbers, and he conjectured that these conditions were also necessary for a covering number to be primitive. Recently, the second author and Daniel White have shown that Sun's conjecture is false by finding infinitely many counterexamples. In this article, we give necessary and sufficient conditions for certain positive integers to be primitive covering numbers. We use these results to answer a question of Sun, and to prove the existence of infinitely many previously-unknown primitive covering numbers. We also show, for each of these new primitive covering numbers L, that a covering can be constructed with distinct moduli using only a proper subset of the divisors $d > 1$ of L as moduli.

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http://dx.doi.org/10.1016/j.jnt.2016.08.017
0022-314X/

1. Introduction

A *covering system*, or simply a *covering*, of the integers is a finite collection of congruences $x \equiv r_i \pmod{m_i}$, such that every integer satisfies at least one of these congruences. The concept of a covering is originally due to Erdős [3].

In 2007, Zhi-Wei Sun [8] introduced the notion of a primitive covering number. A positive integer L is called a *covering number* if there exists a covering of the integers where the moduli are distinct divisors $d > 1$ of L. Clearly, if L is a covering number, then any multiple of L is a covering number. A covering number L is called a *primitive covering number* if no proper divisor of L is a covering number. The following theorem, due to Sun [8], gives sufficient conditions for a positive integer to be a covering number.

Theorem 1.1. *Let $p_1, p_2, \ldots, p_r$ be distinct primes, and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive integers. Suppose that*

$$\prod_{0<t<s} (\alpha_t + 1) \geq p_s - 1 + \delta_{r,s}, \qquad \text{for all } s = 1, 2, \ldots, r, \tag{1.1}$$

where $\delta_{r,s}$ is Kronecker's delta, and the empty product $\prod_{0<t<1}(\alpha_t + 1)$ is defined to be 1. Then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is a covering number.

The next theorem, also due to Sun [8], gives sufficient conditions for a positive integer to be a primitive covering number.

Theorem 1.2. *Let $r > 1$ and let $2 = p_1 < p_2 < \cdots < p_r$ be primes. Suppose further that $p_{t+1} \equiv 1 \pmod{p_t - 1}$ for all $0 < t < r - 1$, and $p_r \geq (p_{r-1} - 2)(p_{r-1} - 3)$. Then*

$$p_1^{\frac{p_2-1}{p_1-1}-1} \cdots p_{r-2}^{\frac{p_{r-1}-1}{p_{r-2}-1}-1} p_{r-1}^{\left\lfloor \frac{p_r-1}{p_{r-1}-1} \right\rfloor} p_r$$

is a primitive covering number, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Theorem 1.2 produces an infinite set of primitive covering numbers, each of which satisfies (1.1). We observe that all primitive covering numbers $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ produced by Theorem 1.2 have $\alpha_1 = p_2 - 2$ and $\alpha_r = 1$.

Concerning Theorem 1.1, Sun [8] made the following conjecture.

Conjecture 1.3. *Any primitive covering number can be written as $p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \ldots, p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers that satisfy (1.1).*

The second author and Daniel White have shown recently that Conjecture 1.3 is false by establishing the following theorem [5]. We let q_n denote the nth prime number.