Journal of Number Theory 172 (2017) 270–286

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Generalized Dedekind sums and equidistribution mod 1

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A R T I C L E I N F O

Article history:
Received 5 January 2016
Received in revised form 25 August 2016
Accepted 25 August 2016
Available online 8 October 2016
Communicated by D. Goss

Keywords:
Automorphic forms
Dedekind eta function, Dedekind sums
Kloosterman sums
Fuchsian group
Equidistribution

A B S T R A C T

Dedekind sums are well-studied arithmetic sums, with values uniformly distributed mod 1. Based on their relation to certain modular forms, Dedekind sums may be defined as functions on the cusp set of $\mathrm{SL}(2,\mathbb{Z})$. We present a compatible notion of Dedekind sums, which we name Dedekind symbols, for any non-cocompact lattice $\Gamma < \mathrm{SL}(2,\mathbb{R})$, and prove the corresponding equidistribution mod 1 result. The latter part builds up on a paper of Vardi, who first connected exponential sums of Dedekind sums to Kloosterman sums.

1. Introduction

In this article, we introduce a function for non-cocompact lattices of $\mathrm{SL}(2,\mathbb{R})$ that relates to, and actually generalizes, the classical Dedekind sums

$$s(a;c) = \sum_{n=1}^{c-1} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right), \qquad (c \in \mathbb{N},\ a \in \mathbb{Z},\ (a,c)=1),$$

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http://dx.doi.org/10.1016/j.jnt.2016.08.003
0022-314X/

where

$$x \mapsto ((x)) \;:=\; \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \qquad (\{x\} = \text{ fractional part of } x \in \mathbb{R})$$

is the odd and periodic "sawtooth" function of expectancy zero.

There is a ubiquitous character to the Dedekind sums, as they appear in a wide range of contexts. The name alone hinges on their relation to the logarithm of the Dedekind η-function

$$\eta(z) \;=\; e\left(\frac{z}{24}\right) \prod_{n \geq 1} (1 - e(nz)) \qquad \left(e(z) = e^{2\pi i z}\right)$$

defined on the upper half-plane $\mathbb{H}$, a classical player in the theories of modular forms, elliptic curves, and theta functions. More precisely, for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$,

$$\log \eta(\gamma z) - \log \eta(z) \;=\; \frac{1}{2}(\mathrm{sign}(c))^2 \log\left(\frac{cz + d}{i \;\mathrm{sign}(c)}\right) + \frac{\pi i}{12}\Phi(\gamma), \tag{1.1}$$

where the defect $\Phi(\gamma)$ arising from the ambiguity of the principal branch of the logarithm is given by

$$\Phi(\gamma) \;=\; \begin{cases} b/d & c = 0, \\ \frac{a+d}{c} - 12\mathrm{sign}(c)s(a; |c|) & c \neq 0. \end{cases} \tag{1.2}$$

While this is not obvious at first glance, the values of Φ are always integers. The latter fact, as many other fundamental properties pertaining to Dedekind sums, may be found in the monograph [RadG72]. Dedekind's original proof of the transformation formula (1.1) is of analytic nature, but it can also be deduced by purely topological arguments. Atiyah [Ati87] discusses this approach and offers an overview of the appearance of $\log \eta$ and the Dedekind sums in various contexts of number theory, topology and geometry, exhibiting no less than seven equivalent characterizations of $\log \eta$ across these different fields!

An alternative presentation of the Dedekind sums consists in defining $s(a; c)$ as a function on the cusp set of $\mathrm{SL}(2, \mathbb{Z})$, which can be identified with the extended rational line $\mathbb{Q} \cup \{\infty\}$. This identification can then be exploited to study some of their properties via continued fraction expansions, as is done in [KM94]. We propose a modified construction. Let Γ_∞ denote the stabilizer subgroup of $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ at ∞, that is,

$$\Gamma_\infty \;=\; \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \right\} \;=\; \pm \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}.$$

There is a one-to-one correspondence between the cusp set of Γ, i.e. $\{\gamma(\infty) : \gamma \in \Gamma\}$ and the quotient Γ/Γ_∞. We can thus express (signed) Dedekind sums via the assignment