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The van der Waerden complex

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ABSTRACT

We introduce the van der Waerden complex $\mathrm{vdW}(n, k)$ defined as the simplicial complex whose facets correspond to arithmetic progressions of length k in the vertex set $\{1, 2, \ldots, n\}$. We show the van der Waerden complex $\mathrm{vdW}(n, k)$ is homotopy equivalent to a CW-complex whose cells asymptotically have dimension at most $\log k / \log \log k$. Furthermore, we give bounds on n and k which imply that the van der Waerden complex is contractible.

1. Introduction

A number of recent papers have considered the topology of cell complexes associated to number theoretic concepts. These include Björner's study of the Betti numbers and homotopy type of a simplicial complex whose Euler characteristic is the negative of the Mertens function, as well as a related CW-complex whose Euler characteristic is the summatory Liouville function [1], Musiker and Reiner's work describing the coefficients

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of the cyclotomic polynomial as the torsion homology of a sequence of simplicial complexes [8], and Pakianathan and Winfree's topological reformulations of number theoretic conjectures using threshold complexes [9].

Recall that van der Waerden's Theorem from Ramsey theory states that given positive integers k and r, there is an integer $M = M(k, r)$ so that when the integers 1 through n with $n \geq M$ are colored with r colors, there is a monochromatic arithmetic progression of length k [11]. There is an upper bound for $M(k, r)$ due to Gowers coming out of his proof of Szemeredi's theorem [3].

Motivated by van der Waerden's theorem, we define the *van der Waerden complex* $\mathrm{vdW}(n, k)$ to be the simplicial complex on the vertex set $\{1, 2, \ldots, n\}$ whose facets correspond to all arithmetic progressions of length k, that is, the facets have the form $\{x, x + d, x + 2 \cdot d, \ldots, x + k \cdot d\}$, where d is a positive integer and $1 \leq x < x + k \cdot d \leq n$.

Observe that the van der Waerden complex $\mathrm{vdW}(n, k)$ is a simplicial complex of dimension k, that is, each facet has dimension k. Furthermore, when $k = 1$, the complex $\mathrm{vdW}(n, 1)$ is the complete graph K_n, which is homotopy equivalent to a wedge of $\binom{n-1}{2}$ circles.

This paper is concerned with understanding the topology of the van der Waerden complex. By constructing a discrete Morse matching, we show that the dimension of the homotopy type of the van der Waerden complex is bounded above by the maximum of the number of distinct primes factors of all positive integers less than or equal to k. See Theorem 3.8. This bound is asymptotically described as $\log k/\log\log k$. See Theorem 3.11. In Section 4 we give bounds under which the van der Waerden complex is contractible. We then look at the implications of our results when studying the topology of the family of van der Waerden complexes $\mathrm{vdW}(5k, k)$ where k is any positive integer. We end with open questions in the concluding remarks.

2. Preliminaries

Let $\mathbb{P}$ be the set of positive integers. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $[i, j]$ denote the interval $\{i, i + 1, \ldots, j\}$.

Definition 2.1. The *van der Waerden complex* $\mathrm{vdW}(n, k)$ is the simplicial complex on the vertex set $[n]$ whose facets correspond to all arithmetic progressions of length k in $[n]$, that is, the facets have the form

$$\{x, x + d, x + 2 \cdot d, \ldots, x + k \cdot d\},$$

where d is a positive integer and $1 \leq x < x + k \cdot d \leq n$.

We remark that the van der Waerden complex is not monotone in the variable k. For instance, the set $\{1, 4, 7\}$ is a facet in $\mathrm{vdW}(7, 2)$, but is not a face of $\mathrm{vdW}(7, 3)$.

Let P be a finite partially ordered set (poset) with partial order relation $\prec$. For further information on posets, see [10, Chapter 3]. A *matching* M on P is a collection of disjoint