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General solutions of sums of consecutive cubed integers equal to squared integers

Vladimir Pletser

European Space Research and Technology Centre, ESA-ESTEC, P.O. Box 299, NL-2200 AG Noordwijk, The Netherlands

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A B S T R A C T

All integer solutions (M, a, c) to the problem of the sums of M consecutive cubed integers $(a + i)^3$ $(a > 1,\ 0 \leq i \leq M - 1)$ equaling squared integers c^2 are found by decomposing the product of the difference and sum of the triangular numbers of $(a + M - 1)$ and $(a - 1)$ in the product of their greatest common divisor g and remaining square factors δ^2 and σ^2, yielding $c = g\delta\sigma$. Further, the condition that g must be integer for several particular and general cases yields generalized Pell equations whose solutions allow to find all integer solutions (M, a, c) showing that these solutions appear recurrently. In particular, it is found that there always exists at least one solution for the cases of all odd values of M, of all odd integer square values of a, and of all even values of M equal to twice an integer square.

E-mail address: Vladimir.Pletser@esa.int.

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1. Introduction

It is known since long that the sum of M consecutive cubed positive integers starting from 1 equals the square of the sum of the M consecutive integers, which itself equals the triangular number $\triangle_M$ of the number of terms M,

$$\sum_{i=1}^{M} i^3 = \left(\sum_{i=1}^{M} i\right)^2 = \left(\frac{M\,(M+1)}{2}\right)^2 = \triangle_M^2,$$

for $\forall M \in \mathbb{Z}^+$. The question whether this remarkable result can be extended to other integer values of the starting point, i.e. whether the sum of consecutive cubed positive integers starting from $a \neq 1$ is also a perfect square,

$$\sum_{i=0}^{M-1} (a+i)^3 = c^2, \tag{1.1}$$

has been addressed by several authors but has received so far only partial answers.

With the notation of this paper, Lucas stated [15] that the only solutions for $M = 5$ are $a = 0$, 1, 96 and 118 (missing the solution $a = 25$, see further Table 1), and that there are no other solutions for $M = 2$ than $a = 1$. Aubry showed [1] that a solution for $M = 3$ is $a = 23$, $c = 204$, correcting Lucas' statement that there are no other solution for $M = 3$ than $a = 1$. Other historical accounts can be found in [7]. Cassels proved [3] by using the method of finding all integral points on a given curve of genus 1 $y^2 = 3x\left(x^2+2\right)$ (with $x = a + 1$, $y = c$ in this paper notations), that the only solutions for $M = 3$ are $a = -1$, 0, 1 and 23. Stroeker obtained [31] complete solutions for $2 \leq M \leq 50$ and $M = 98$, using estimates of lower bound of linear forms in elliptic logarithms to solve elliptic curve equations of the form $Y^2 = X^3 + dX$ where $d = n^2\left(n^2-1\right)/4$, $X = nx + n\,(n-1)\,/2$, $Y = ny$ (with $n = M$, $x = a$, $y = c$ in this paper notations). The method reported, although powerful, appears long and difficult and caused some problems for the cases $M = 41$ and 44. Stroeker remarked also that $M = a = 33$ with $c = 2079 = 33 \times 63$. This is not the single occurrence of $M = a$, as it occurs also for $M = a = 2017, 124993, 7747521, \ldots$ (see Sequences A180920 and A180921 in [24]).

One of the reasons that these previous attempts to find all solutions (M, a, c) to (1.1) were only partially successful was most likely due to the approach taken to start the search for solutions for single values of M, one by one and in an increasing order of M values. The method proposed in this paper is to tackle the problem in a different way and instead of looking at each individual value of M one by one, to consider the problem in a more global approach by comparing different sets of known solutions, and instead of listing solutions in increasing order of M values, to look at two other parameters, δ and σ, defined further. This new beginning then leads to a more classical approach using general solutions of Pell equations, that allows to find all solutions in (M, a, c) of (1.1) for all possible cases. Note that Pell equations were already used previously by various authors