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Simple transitive 2-representations for some 2-subcategories of Soergel bimodules

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ABSTRACT

We classify simple transitive 2-representations of certain 2-subcategories of the 2-category of Soergel bimodules over the coinvariant algebra in Coxeter types B_2 and $I_2(5)$. In the $I_2(5)$ case it turns out that simple transitive 2-representations are exhausted by cell 2-representations. In the B_2 case we show that, apart from cell 2-representations, there is a unique, up to equivalence, additional simple transitive 2-representation and we give an explicit construction of this 2-representation.

1. Introduction and description of the results

Classification problems are, historically, the main driving force of representation theory. The desire to understand and, in particular, classify certain classes of representations of a given group or algebra was behind the majority of research in the general area of representation theory since its birth.

The abstract 2-representation theory, which originated in [3,5,10,23], studies functorial actions of 2-categories. The "finite-dimensional" part of this theory, that is 2-representation theory of finitary 2-categories, was systematically developed in the series [16–21] and continued in [27–29]. In particular, the paper [20] proposes a very good candidate for the notion of a "simple" 2-representation, called a *simple transitive* 2-representation. In the same paper one finds a classification of such 2-representations for a special class of finitary 2-categories with involution which includes the 2-category of Soergel bimodules over the coinvariant algebra of the symmetric group. This is extended in [21] to more general 2-categories and (slightly) more general classes of 2-representations. Some nice applications of these classification results were obtained in [11].

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The classification of simple transitive 2-representations for some "smallest" 2-categories which do not fit the setup and methods of [20,21] was completed in [22,30]. All the results mentioned above have, however, one common feature. It turns out that in all these cases the simple transitive 2-representations are exhausted by the so-called *cell 2-representations* defined and studied in [16]. So far there was only one, quite artificial, example of a family of simple transitive 2-representations which are not equivalent to cell 2-representations, constructed in [20, Subsection 3.2] using transitive group actions.

In the present paper we study simple transitive 2-representations of a certain 2-subquotient $\mathscr{Q}_n$ of the 2-category of Soergel bimodules for the dihedral group $D_{2\cdot n}$, where $n \geq 3$. For $n = 3$, this 2-category fits into the setup of [20] and hence the classification result from [20] directly applies. For $n > 3$, the 2-category $\mathscr{Q}_n$ must be studied by other methods.

We show that every simple transitive 2-representation of $\mathscr{Q}_5$ is equivalent to a cell 2-representation, see Theorem 5. We also show that, apart from cell 2-representations, there is a unique, up to equivalence, simple transitive 2-representation of $\mathscr{Q}_4$, see Theorem 12. The corresponding 2-representation is explicitly constructed in Subsection 5.8. This subsection is the heart of this paper. Construction of this new 2-representation is based on the careful interplay of several category-theoretic tricks. The case $n > 5$ seems, at the moment, computationally too difficult.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries from 2-representation theory. In Section 3 we recall the definition and combinatorics of the 2-category of Soergel bimodules over a dihedral group and define the 2-category $\mathscr{Q}_n$, our main object of study. Theorem 5 is proved in Section 4. Theorem 12 is proved in Section 5.

2. Generalities on 2-categories and 2-representations

2.1. Notation and conventions

We work over $\mathbb{C}$ and write $\otimes$ for $\otimes_{\mathbb{C}}$. By a module we mean a *left* module. Maps are composed from right to left.

2.2. Finitary and fiat 2-categories

We refer the reader to [13–15] for generalities on 2-categories.

A 2-category is a category enriched over the monoidal category **Cat** of small categories. Thus, a 2-category $\mathscr{C}$ consists of objects (denoted by Roman lower case letters in a typewriter font), 1-morphisms (denoted by capital Roman letters), and 2-morphisms (denoted by Greek lower case letters), composition of 1-morphisms, horizontal and vertical compositions of 2-morphisms (denoted $\circ_0$ and $\circ_1$ respectively), identity 1-morphisms and identity 2-morphisms. These must satisfy the obvious collection of axioms. For a 1-morphism F, we denote by id_{F} the corresponding identity 2-morphism. As usual, we often write $\mathrm{F}(\alpha)$ for $\mathrm{id}_{\mathrm{F}} \circ_0 \alpha$ and α_{F} for $\alpha \circ_0 \mathrm{id}_{\mathrm{F}}$.

A 2-category $\mathscr{C}$ is called *finitary* if, for each pair $(\mathtt{i}, \mathtt{j})$ of objects in $\mathscr{C}$, the category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is an idempotent split, additive and Krull–Schmidt $\mathbb{C}$-linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces; moreover, all compositions must be compatible with these additional structures, see [16] for details.

A finitary 2-category $\mathscr{C}$ is called *fiat* if it has a weak involution $\star$ together with adjunction 2-morphisms satisfying the usual axioms of adjoint functors, for each pair $(\mathrm{F}, \mathrm{F}^{\star})$ of 1-morphisms, see [16] for details.