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Ratliff–Rush filtration, regularity and depth of higher associated graded modules. Part II

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ABSTRACT

Let $(A, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated Cohen–Macaulay A-module of dimension $r \geq 2$ and let I be an ideal of definition for M. Set $L^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. In part one of this paper we showed that $L^I(M)$ is a module over $\mathcal{R}(I)$, the Rees algebra of I and we gave many applications of $L^I(M)$ to study the associated graded module, $G_I(M)$. In this paper we give many further applications of our technique; most notable is a complete characterization of good behavior of the Ratliff–Rush filtration modulo a superficial element.

0. Introduction

Dear Reader, while reading this paper it is a good idea to have part 1 of this paper [11]. Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $k = A/\mathfrak{m}$. Let M be a finitely generated Cohen–Macaulay A-module of dimension $r \geq 2$ and let I be an ideal of definition for M i.e., $\lambda(M/IM)$ is finite. Here $\lambda(-)$ denotes length. Let $G_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of A with respect to I and let $G_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$ be the associated graded module of M with respect to I.

Set $L^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. In part one of this paper we showed that $L^I(M)$ is a module over $\mathcal{R}(I)$, the Rees-algebra of I. It is *not finitely generated* as a $\mathcal{R}(I)$-module. In part 1 we gave applications of $L^I(M)$ in the study of associated graded modules.

0.1. Applications

In part 1 of this paper we gave *five* applications of the technique of $L^I(M)$ in the study of $G_I(M)$. In part 2 we give *six* more applications of our technique. If E is an $\mathcal{R}(I)$-module then we set $H^i(E)$ to be the ith-local cohomology module of E with respect to the maximal homogeneous ideal of $\mathcal{R}(I)$.

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VI. For definition of Ratliff–Rush filtration see 1.4. Let x be M-superficial with respect to I. Set $N = M/xM$ and $u = xt \in \mathcal{R}(I)$. We say the Ratliff–Rush filtration on M with respect to I *behaves well mod* x if

$$\overline{\widetilde{I^nM}} = \widetilde{I^nN} \quad \text{for all } n \geq 1.$$

We prove that the Ratliff–Rush filtration on M with respect to I behaves well mod x if and only if $H^1(L^I(M)) = 0$; see Theorem 2.3. In particular our result proves that if Ratliff–Rush filtration behaves well mod one superficial element then it does so *with any* superficial element.

We then relate vanishing of $H^i(L^I(M))$ for $i = 1, \dots, s$ to good behavior of Ratliff–Rush filtration mod a superficial sequence of length s; see Theorem 4.5. Thus good behavior of the Ratliff–Rush filtration mod a superficial sequence is a cohomological property.

VII. *minimal* $\mathbb{I}$*-invariant*:

Recall that we say $G_I(M)$ is generalized Cohen–Macaulay module if

$$\lambda(H^i(G_I(M))) < \infty \text{ for } i = 0, 1, \cdots, r-1.$$

For generalized Cohen–Macaulay module the Stückrad–Vogel invariant

$$\mathbb{I}(G_I(M)) = \sum_{i=0}^{r-1} \binom{r-1}{i} \lambda(H^i(G_I(M)))$$

plays a crucial role. If $x^* \in G_I(A)_1$ is $G_I(M)$-regular then one can verify

$$\mathbb{I}(G_I(M/xM)) \leq \mathbb{I}(G_I(M)).$$

So in some sense if we have to study minimal $\mathbb{I}$-invariant then we have to first consider the case when $\operatorname{depth} G_I(M) = 0$. In Theorem 5.4 we prove that if $G_I(M)$ is generalized Cohen–Macaulay and $\operatorname{depth} G_I(M) = 0$, then

$$\mathbb{I}(G_I(M)) \geq r \cdot \lambda\left(H^0(G_I(M))\right).$$

We also prove that the following are equivalent

(i) $\mathbb{I}(G_I(M)) = r \cdot \lambda(H^0(G_I(M)))$.
(ii) $H^i(L^I(M)) = 0$ for $i = 1, 2, \cdots, r-1$.
(iii) The Ratliff–Rush filtration on M behaves well mod superficial sequences (of length $r-1$).

VIII. A classical result, due to Narita [9] states that if $(A, \mathfrak{m})$ is Cohen–Macaulay of dim 2 then

$$e_2^I(A) = 0 \ \text{ iff } \ \operatorname{red}(I^n) = 1 \text{ for all } n \gg 0.$$

This can be easily extended to Cohen–Macaulay modules of dimension two. However Narita's result fails (even for Cohen–Macaulay rings) in dimension ≥ 3; see 6.3. We first reformulate Narita's result in dimension 2.

Let $\widetilde{G}_I(M) = \bigoplus_{n \geq 0} \widetilde{I^nM}/\widetilde{I^{n+1}M}$ be the associated graded module of the Ratliff–Rush filtration. We prove

$$e_2^I(M) = 0 \Longleftrightarrow \widetilde{G}_I(M) \ \text{ has minimal multiplicity.}$$