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On intersections of two-sided ideals of Leavitt path algebras

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A R T I C L E I N F O

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A B S T R A C T

Let E be an arbitrary directed graph and let L be the Leavitt path algebra of the graph E over a field K. It is shown that every ideal of L is an intersection of primitive/prime ideals in L if and only if the graph E satisfies Condition (K). Uniqueness theorems in representing an ideal of L as an irredundant intersection and also as an irredundant product of finitely many prime ideals are established. Leavitt path algebras containing only finitely many prime ideals and those in which every ideal is prime are described. Powers of a single ideal I are considered and it is shown that the intersection $\bigcap_{n=1}^{\infty} I^n$ is the largest graded ideal of L contained in I. This leads to an analogue of Krull's theorem for Leavitt path algebras.

1. Introduction and preliminaries

Leavitt path algebras $L_K(E)$ of directed graphs E over a field K are algebraic analogues of graph C^*-algebras $C^*(E)$ and have recently been actively investigated in a series of papers (see e.g. [2–4,8,11]). These investigations showed, in a number of cases, how an algebraic property of $L_K(E)$ and the corresponding analytical property of $C^*(E)$ are both implied by the same graphical property of E, though the techniques of proofs are often different. The initial investigation of special types of ideals such as the graded ideals, the corresponding quotient algebras and the prime ideals of a Leavitt path algebra was essentially inspired by the analogous investigation done for graph C^*-algebras. But an extensive investigation of the ideal theory of Leavitt path algebras, as has been done for commutative rings, is yet to happen.

This paper may be considered as a small step in exploring the multiplicative ideal theory of Leavitt path algebras and was triggered by a question raised by Astrid an Huef. In any C^*-algebras, and in any graph C^*-algebra in particular, every (closed) ideal is the intersection of all the primitive/prime ideals containing it

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(Theorem 2.9.7, [7]). In a recent 2015 CIMPA research school in Turkey on Leavitt path algebras and graph C^*-algebras, Astrid an Huef raised the question whether the preceding statement is true for ideals of Leavitt path algebras. We first construct examples showing that this property does not hold in general for Leavitt path algebras. We then prove that, for a given graph E, every ideal of the Leavitt path algebra $L_K(E)$ is an intersection of primitive/prime ideals if and only if the graph E satisfies Condition (K). A uniqueness theorem is proved in representing an ideal of $L_K(E)$ as the irredundant intersection of finitely many prime ideals. Recall that an intersection $P_1 \cap \ldots \cap P_m$ of ideals is *irredundant* if no P_i contains the intersection of the other $m-1$ ideals P_j, $j \neq i$. As a corollary, we show that every ideal of $L_K(E)$ is a prime ideal if and only if (i) Condition (K) holds in E, (ii) for each hereditary saturated subset H of vertices, $|B_H| \leq 1$ and $E^0 \backslash H$ is downward directed and (iii) the admissible pairs (H, S) (see definition below) form a chain under a defined partial order. Equivalently, all the ideals of $L_K(E)$ are graded and form a chain under set inclusion. Following this, Leavitt path algebras possessing finitely many prime ideals are described. We also give conditions under which every ideal of a Leavitt path algebra is an intersection of maximal ideals.

The graded ideals of a Leavitt path algebra possess many interesting properties. Using these, we examine the uniqueness of factorizing a graded ideal as a product of prime ideals. Recall that $I = P_1 \cdots P_n$ is an **irredundant product** of the ideals P_i, if I is not the product of a proper subset of this set of n ideals P_i. An interesting result is that if I is a graded ideal and $I = P_1 \cdots P_n$ is a factorization of I as an irredundant product of prime ideals P_i, then necessarily all the ideals P_i must be graded ideals and moreover, $I = P_1 \cap \ldots \cap P_n$. We also prove a weaker version of this result for non-graded ideals. Finally, powers of an ideal in $L_K(E)$ are studied. While $I^2 = I$ for any graded ideal I, it is shown that, for a non-graded ideal I of $L_K(E)$, its powers I^n $(n \geq 1)$ are all non-graded and distinct, but the intersection of the powers $\bigcap_{n=1}^{\infty} I^n$ is always a graded ideal and is indeed the largest graded ideal of $L_K(E)$ contained in I. As a corollary, we obtain an analogue of Krull's theorem (Theorem 12, section 7, [12]) for Leavitt path algebras: For an ideal I of $L_K(E)$, the intersection $\bigcap_{n=1}^{\infty} I^n = 0$ if and only if I contains no vertices.

Preliminaries: For the general notation, terminology and results in Leavitt path algebras, we refer to [2,8] and [11]. For basic results in associative rings and modules, we refer to [5]. We give below a short outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular vertex* if it emits a non-empty finite set of edges. An *infinite emitter* is a vertex which emits infinitely many edges. For each $e \in E^1$, we call e^* a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A *path* μ of length $|\mu| = n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n-1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path. A vertex is considered a path of length 0. The set of all vertices on the path μ is denoted by μ^0.

A path $\mu = e_1 \cdots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, ..., n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

An *exit* for a path $\mu = e_1 \cdots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$. We say the graph E satisfies *Condition (L)* if every cycle in E has an exit. The graph E is said to satisfy *Condition (K)* if every vertex which is the base of a closed path c is also a base of another closed path c' different from c.

If there is a path from vertex u to a vertex v, we write $u \geq v$. A subset D of vertices is said to be *downward directed* if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is *saturated* if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.