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Tensor products of higher almost split sequences

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ABSTRACT

We investigate how the higher almost split sequences over a tensor product of algebras are related to those over each factor. Herschend and Iyama give in [6] a criterion for when the tensor product of an n-representation finite algebra and an m-representation finite algebra is $(n+m)$-representation finite. In this case we give a complete description of the higher almost split sequences over the tensor product by expressing every higher almost split sequence as the mapping cone of a suitable chain map and using a natural notion of tensor product for chain maps.

1. Introduction and conventions

In the context of Auslander–Reiten theory one can study almost split sequences of modules over a finite-dimensional algebra A. These are certain short exact sequences

$$0 \to M \to N \to L \to 0$$

such that M and L are indecomposable, and it turns out that every nonprojective indecomposable module over A appears as the last term of such a sequence (and every noninjective indecomposable appears as the first term). Moreover, such sequences are determined up to isomorphism by either the first or the last term (see for reference [2]). One can do a similar construction in the context of higher dimensional Auslander–Reiten theory, at the cost of restricting to a suitable subcategory $\mathcal{C}$ of mod A that contains all injectives and all projectives. Then one gets longer so called n-almost split sequences

$$0 \to M \to X_1 \to \cdots \to X_n \to L \to 0$$

in $\mathcal{C}$, and again every nonprojective module in $\mathcal{C}$ appears at the end of such a sequence and every noninjective at the start of one. Again, these sequences are determined by their first or last term (see [8,9]). One of the most basic cases where such a situation appears is when A is n-representation finite (cf. [6,8]).

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Definition. Let A be a finite-dimensional k-algebra, and let $n \in \mathbb{Z}_{>0}$. An *n-cluster tilting module* for A is a module $M_A \in \operatorname{mod} A$ such that

$$\begin{aligned}\operatorname{add} M_A &= \left\{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(M_A, X) = 0 \text{ for every } 0 < i < n\right\} = \\ &= \left\{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(X, M_A) = 0 \text{ for every } 0 < i < n\right\}.\end{aligned}$$

We say that A is *n-representation finite* if $\operatorname{gl.dim} A \leq n$ and there exists an n-cluster tilting module for A. Then $\operatorname{gl.dim} A = 0$ or $\operatorname{gl.dim} A = n$.

For such algebras it is known that $\operatorname{add} M_A$ is a subcategory of $\operatorname{mod} A$ that admits n-almost split sequences. We call D the functor $D = \operatorname{Hom}_k(-, k) : \operatorname{mod} A \to A \operatorname{mod}$. The (higher) *Auslander–Reiten translations* τ_n, τ_n^- are defined as follows:

$$\tau_n = D \operatorname{Ext}^n_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A$$

$$\tau_n^- = \operatorname{Ext}^n_A(DA, -) : \operatorname{mod} A \to \operatorname{mod} A.$$

It is immediate from this definition that

$$\tau_n A = 0 = \tau_n^- DA.$$

These higher Auslander–Reiten translations behave similarly to the classical ones.

Theorem. *Let A be an n-representation finite k-algebra. Let $P_1, \ldots, P_a$ be nonisomorphic representatives of the isomorphism classes of indecomposable projective right A-modules, and $I_1, \ldots, I_a$ the corresponding indecomposable injective modules. Then:*

(1) *There exist positive integers $l_1, \ldots, l_a$ and a permutation $\sigma \in S_a$ (the symmetric group over a elements) such that $P_i \cong \tau_n^{l_i - 1} I_{\sigma(i)}$ for every i.*
(2) *There exists a unique (up to isomorphism) basic n-cluster tilting module M_A, which is given by*

$$M_A = \bigoplus_{i=1}^{a} \bigoplus_{j=0}^{l_i - 1} \tau_n^j I_{\sigma(i)}.$$

(3) *The Auslander–Reiten translations induce mutually quasi-inverse equivalences*

$$\operatorname{add}(M_A/P) \underset{\tau_n}{\overset{\tau_n^-}{\rightleftarrows}} \operatorname{add}(M_A/I)$$

where $P = \bigoplus_{i=1}^a P_i$ and $I = \bigoplus_{i=1}^a I_i$.

Proof. See [9, 1.3(b)]. □

From the last point it follows in particular that the n-cluster tilting module can be equally described by

$$M_A = \bigoplus_{i=1}^{a} \bigoplus_{j=0}^{l_i - 1} \tau_n^{-j} P_i.$$