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Riordan trees and the homotopy sl_2 weight system

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ABSTRACT

The purpose of this paper is twofold. On one hand, we introduce a modification of the dual canonical basis for invariant tensors of the 3-dimensional irreducible representation of $U_q(sl_2)$, given in terms of Jacobi diagrams, a central tool in quantum topology. On the other hand, we use this modified basis to study the so-called homotopy sl_2 weight system, which is its restriction to the space of Jacobi diagrams labeled by distinct integers. Noting that the sl_2 weight system is completely determined by its values on trees, we compute the image of the homotopy part on connected trees in all degrees; the kernel of this map is also discussed.

1. Introduction

The sl_2 weight system W is a $\mathbb{Q}$-algebra homomorphism from the space $\mathcal{B}(n)$ of Jacobi diagrams labeled by $\{1, \dots, n\}$ to the algebra $\mathrm{Inv}\,(S(sl_2)^{\otimes n})$ of invariant tensors of the symmetric algebra $S(sl_2)$. The relevance of this construction lies in low dimensional topology. Jacobi diagrams form the target space for the Kontsevich integral Z, which is universal among finite type and quantum invariants of knotted objects: in particular, by postcomposing Z with the sl_2 weight system and specializing each factor at some finite-dimensional representation of quantum group $U_q(sl_2)$, one recovers the colored Jones polynomial. Hence, while the results of this paper are purely algebraic, we will see that they are motivated by, and have applications to, quantum topology – see Remark 1.4 at the end of this introduction.

An easy preliminary observation on the sl_2 weight system is the following.

Lemma 1.1. *The sl_2 weight system is determined by its values on connected trees, i.e. connected and simply connected Jacobi diagrams.*

(Although this result might be well-known, a proof is given in Section 2.4.)

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Table 1
The dimensions of $\mathcal{C}_n$, $\mathrm{Inv}(sl_2^{\otimes n})$ and $\mathrm{Ker}\, W_n^h$.

n	2	3	4	5	6	7	8	9	k
$\dim \mathcal{C}_n$	1	1	2	6	24	120	720	5040	$(k-2)!$
$\dim \mathrm{Inv}(sl_2^{\otimes n})$	1	1	3	6	15	36	91	232	R_k
$\dim \mathrm{Ker}\, W_n^h$	0	0	0	0	10	84	630	4808	$(k-2)! - R_k + \frac{1+(-1)^k}{2}$

In this paper, we focus on the *homotopy part* $\mathcal{B}^h(n)$ of $\mathcal{B}(n)$, which is generated by diagrams labeled by distinct elements in $\{1, \dots, n\}$. Here, the terminology alludes to the link-homotopy relation on (string) links, which is generated by self crossing changes. It was shown by Habegger and Masbaum [4] that the restriction of the Kontsevich integral to $\mathcal{B}^h(n)$ is a link-homotopy invariant, and is deeply related to Milnor link-homotopy invariants, which are classical invariants generalizing the linking number.

Let us state our main results on the *homotopy sl_2 weight system*, that is, the restriction of the sl_2 weight system to $\mathcal{B}^h(n)$. Owing to Lemma 1.1, we can fully understand this map by studying the restrictions

$$W_n^h \colon \mathcal{C}_n \to \mathrm{Inv}(sl_2^{\otimes n})$$

of the sl_2 weight system to the space $\mathcal{C}_n$ of connected trees with n univalent vertices labeled by distinct elements in $\{1, \dots, n\}$. Here, the target space $\mathrm{Inv}(sl_2^{\otimes n})$ is the invariant part of the n-fold tensor power of the adjoint representation (the 3-dimensional irreducible representation) of sl_2. Recall that the dimension of $\mathcal{C}_n$ is given by $(n-2)!$, while the dimension of $\mathrm{Inv}(sl_2^{\otimes n})$ is known to be the so-called [1] Riordan numbers R_n which can be defined by $R_2 = R_3 = 1$ and $R_n = (n-1)(2R_{n-1} + 3R_{n-2})/(n+1)$. These numbers are also found under the name of Motzkin sums, or ring numbers in the literature.

We have:

Theorem 1.2.

(i) *The weight system map W_n^h is injective if and only if $n \le 5$.*
(ii) *For n odd and $n = 2$, the weight system map W_n^h is surjective.*
(iii) *For $n \ge 4$ even, W_n^h has a 1-dimensional cokernel, spanned by $c^{\otimes \frac{n}{2}}$, where $c = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in \mathrm{Inv}(sl_2^{\otimes 2})$.*

The dimensions of $\mathcal{C}_n$, $\mathrm{Inv}(sl_2^{\otimes n})$ and $\mathrm{Ker}\, W_n^h$ are given in Table 1.

Let $\mathfrak{S}_n$ be the symmetric group in n elements. The spaces $\mathcal{C}_n$ and $\mathrm{Inv}(sl_2^{\otimes n})$ have $\mathfrak{S}_n$-module structures, such that $\mathfrak{S}_n$ acts on $\mathcal{C}_n$ by permuting the labels, and acts on $\mathrm{Inv}(sl_2^{\otimes n})$ by permuting the factors. The sl_2 weight system is a $\mathfrak{S}_n$-module homomorphism, and the characters $\chi_{\mathcal{C}_n}$ and $\chi_{\mathrm{Inv}(sl_2^{\otimes n})}$ are already known (see Lemma 3.7 and Proposition 3.8). Thus, by Theorem 1.2, we can determine the characters $\chi_{\ker(W_n^h)}$ and $\chi_{\mathrm{Im}(W_n^h)}$ of the kernel and the image of W_n^h, respectively, as follows.

Corollary 1.3. (i) *For $n = 2$ or $n > 2$ odd, we have*

$$\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})} \quad \textit{and} \quad \chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})}.$$

(ii) *For $n \ge 4$ even, we have*

$$\chi_{\ker(W_n^h)} = \chi_{\mathcal{C}_n} - \chi_{\mathrm{Inv}(sl_2^{\otimes n})} + \chi_U \quad \textit{and} \quad \chi_{\mathrm{Im}(W_n^h)} = \chi_{\mathrm{Inv}(sl_2^{\otimes n})} - \chi_U,$$

where U is the trivial representation.