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Generators of truncated symmetric polynomials

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ABSTRACT

Adem and Reichstein introduced the ideal of truncated symmetric polynomials to present the permutation invariant subring in the cohomology of a finite product of projective spaces. Building upon their work, I describe a generating set of the ideal of truncated symmetric polynomials in arbitrary positive characteristic, and offer a conjecture for minimal generators.

1. Introduction

Let $\mathbb{F}$ be a field, and let $R = \mathbb{F}[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in $\mathbb{F}$. The symmetric group $\mathfrak{S}_n$ acts on R by permuting the variables. Denote by $R^{\mathfrak{S}_n}$ the invariant subring, i.e., the ring of symmetric polynomials. The ideal of truncated symmetric polynomials in $R^{\mathfrak{S}_n}$ is defined by

$$I_{n,d} = (x_1^{d+1}, \dots, x_n^{d+1})R \cap R^{\mathfrak{S}_n}.$$

The ideal of truncated symmetric polynomials was introduced by A. Adem and Z. Reichstein [1] in the following geometric context. Let $\mathbb{C}P^d$ be the complex projective d-space, and let $BU(n)$ be the classifying space of the unitary group. The symmetric group $\mathfrak{S}_n$ acts on the n-fold product $(\mathbb{C}P^d)^n$ by permuting the factors. Consider the induced action on the cohomology ring $H^*((\mathbb{C}P^d)^n, \mathbb{F})$ and let $H^*((\mathbb{C}P^d)^n, \mathbb{F})^{\mathfrak{S}_n}$ be the invariant subring. The two authors show there is a map $(\mathbb{C}P^d)^n \to BU(n)$ such that the induced map on cohomology restricts to a ring epimorphism

$$H^*(BU(n), \mathbb{F}) \longrightarrow H^*((\mathbb{C}P^d)^n, \mathbb{F})^{\mathfrak{S}_n}.$$

The cohomology ring of $BU(n)$ can be identified with $R^{\mathfrak{S}_n}$, and the kernel of this map can be identified with $I_{n,d}$. Therefore $R^{\mathfrak{S}_n}/I_{n,d}$ and $H^*((\mathbb{C}P^d)^n, \mathbb{F})^{\mathfrak{S}_n}$ are isomorphic as ungraded rings (they become isomorphic

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as graded rings if the grading on $R^{\mathfrak{S}_n}/I_{n,d}$ is stretched out by a factor of 2). This result was used in the same paper to compute the cohomology of the homotopy fiber for a natural fibration over $BU(n)$.

Adem and Reichstein determined a set of generators of $I_{n,d}$ when the characteristic of $\mathbb{F}$ is zero and when it is strictly bigger than $(n+1)/2$. Independently, A. Conca, C. Krattenthaler and J. Watanabe [2] identified generators of $I_{n,d}$ working over the complex numbers (although their methods hold over any field of characteristic strictly bigger than n). In addition, the ideal of truncated symmetric polynomials has generated interest in connection to other topics, such as Lefschetz properties [5] and invariants of Poincaré duality algebras [8].

In this paper, I describe a generating set of $I_{n,d}$ in arbitrary characteristic. Given a partition λ, let $m_\lambda \in R^{\mathfrak{S}_n}$ denote the monomial symmetric polynomial associated with λ. The shorthand notation $(\dots, a^b, \dots)$ denotes a partition with the part a repeated b times. The main result reads as follows.

Theorem. *Let $t = \max\{i \in \mathbb{N} \mid p^i \leqslant n\}$ and let $q_0, \dots, q_t \in \mathbb{N}_{>0}$ be such that, $\forall i \in \{0, \dots, t\}$, $n = q_i p^i + r_i$ with $0 \leqslant r_i < p^i$. For every $i \in \{0, \dots, t\}$, define an ideal of $R^{\mathfrak{S}_n}$*

$$J_{n,d,i} = (m_{((d+1)^{p^i})}, \dots, m_{((d+q_i)^{p^i})}).$$

Then $I_{n,d} = J_{n,d,0} + J_{n,d,1} + \dots + J_{n,d,t}$.

The next section contains a brief review of symmetric polynomials and some computational results. In section 3, I review the algebraic results of Adem and Reichstein. The proof of the main theorem is in section 4. Finally, in section 5, I present a conjecture for a minimal generating set of the ideal of truncated symmetric polynomials.

2. Symmetric polynomials and partitions

Let $\mathbb{F}$ be a field of characteristic p. For $n \in \mathbb{N}_{>0}$, set $R = \mathbb{F}[x_1, \dots, x_n]$, the polynomial ring in n indeterminates over $\mathbb{F}$. The symmetric group $\mathfrak{S}_n$ acts on R by permuting the variables. Let $R^{\mathfrak{S}_n} \subseteq R$ be the invariant subring, i.e., the ring of symmetric polynomials.

Recall that a *partition* is a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots)$$

of non-negative integers in non-increasing order:

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r \geqslant \dots$$

and containing only finitely many nonzero terms. We identify two such sequences which differ only by a string of zeroes at the end. The nonzero numbers λ_i are called the *parts* of λ; the number of parts of λ is called the *length* of λ and denoted $l(\lambda)$.

For a partition λ with $l(\lambda) \leqslant n$, define the *monomial symmetric polynomial* on λ to be the polynomial,

$$m_\lambda = \sum x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

summed over all distinct permutations $\alpha = (\alpha_1, \dots, \alpha_n)$ of $\lambda = (\lambda_1, \dots, \lambda_n)$. The polynomials m_λ are clearly symmetric. Moreover, if $\mathcal{L}_{\leqslant n}$ denotes the set of all partitions with length smaller than or equal to n, then $\{m_\lambda \mid \lambda \in \mathcal{L}_{\leqslant n}\}$ is a basis of $R^{\mathfrak{S}_n}$ as an $\mathbb{F}$-vector space [6, §I.2, pp. 18–19]. If $l(\lambda) > n$, then $m_\lambda = 0$.