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Witt, GW, K-theory of quasi-projective schemes

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ABSTRACT

In this article, we prove some results on Witt, Grothendieck–Witt (GW) and K-theory of noetherian quasi-projective schemes X, over affine schemes $\mathrm{Spec}\,(A)$. For integers $k \geq 0$, let $C\mathbb{M}^k(X)$ denote the category of coherent $\mathcal{O}_X$-modules $\mathcal{F}$, with locally free dimension $\dim_{\mathscr{V}(X)}(\mathcal{F}) = k = grade(\mathcal{F})$. We prove that there is an equivalence $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathscr{D}^k\left(\mathscr{V}(X)\right)$ of the derived categories. It follows that there is a sequence of zig-zag maps $\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^k(X)\right) \longrightarrow \coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^k(X_x)\right)$ of the $\mathbb{K}$-theory spectra that is a homotopy fibration. In fact, this is analogous to the homotopy fiber sequence of the G-theory spaces of Quillen (see proof of [16, Theorem 5.4]). We also establish similar homotopy fibrations of **GW**-spectra and $\mathbb{G}W$-bispectra, by application of the same equivalence theorem.

1. Introduction

In [16], Quillen established the foundation of K-theory of regular schemes X in a complete manner. In fact, for any scheme X, Quillen provides a complete foundation of K-theory of the category $Coh(X)$ of the coherent sheaves on X, along with that of the filtration of $Coh(X)$ by co-dimension of support of the objects $\mathcal{F} \in Coh(X)$. This relates to Gersten complexes and spectral sequences associated to any such scheme X (see [16, §5]). The K-theory of $Coh(X)$ is also known as G-theory. For regular schemes X, the K-theory of the category $\mathscr{V}(X)$ of locally free sheaves agrees fully with that of $Coh(X)$. Consequently, the K-theory of regular schemes appears very complete. However, the K-theory of non-regular schemes never reached the completeness and harmony that the K-theory of regular schemes had achieved. Work of Waldhausen [23] and Thomason–Trobaugh [22] would be milestones in this respect, most notably for their introduction of derived invariance theorems and localization theorems, applicable to non-regular schemes. Further, while developments in Grothendieck–Witt theory (GW-theory) and Witt theory followed the foot prints of K-theory [20,1], due to the lack of any natural duality on $Coh(X)$, the situation in these two areas appears even less complete. When X is non-regular, the category $\mathbb{M}(X)$ of coherent sheaves with finite

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$\mathscr{V}(X)$-dimension differs from $Coh(X)$. There appears to be a gap in the literature of K-theory, GW-theory, and Witt theory, with respect to the place of the category $\mathbb{M}(X)$. One can speculate, whether this lack of completeness is attributable to this gap. The goal of this one and the related articles is to work on this gap and attempt to establish the said literature on non-regular schemes at the same pedestal as that of regular schemes. For quasi-projective schemes over noetherian affine schemes, this goal is accomplished up to some degree of satisfaction. The special place of the full subcategory $C\mathbb{M}^k(X) \subseteq \mathbb{M}(X)$ would also be clear subsequently, where for integers $k \geq 0$, $C\mathbb{M}^k(X)$ will denote the full subcategory of objects $\mathcal{F}$ in $\mathbb{M}(X)$, with $\dim_{\mathscr{V}(X)}(\mathcal{F}) = grade(\mathcal{F}) = k$.

With respect to certain facets of Algebraic K-theory, Grothendieck–Witt (GW) theory and Witt theory, a common thread among them is their invariance properties with respect to equivalences of the associated Derived categories. We review some of the results on such invariances. For example, recall the theorem of Thomason–Trobaugh [22, Theorem 1.9.8]: suppose $\mathbf{A} \to \mathbf{B}$ is a functor of complicial exact categories with weak equivalences. Assume that the associated functor of the triangulated categories $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ is an equivalence. Then, the induced map $\mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B})$ of the $\mathbf{K}$-theory spaces is a homotopy equivalence (see [18, 3.2.24]). The non-connective version of this theorem was given by Schlichting ([19, Theorem 9], also see [18, 3.2.29]) which states, under the relaxed hypothesis, that: if $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ is an equivalence up to factors, then it induces a homotopy equivalence $\mathbb{K}(\mathbf{A}) \to \mathbb{K}(\mathbf{B})$ of the $\mathbb{K}$-theory spectra. While K-theory is defined for complicial exact categories with weak equivalences, Schlichting defined Grothendieck Witt (GW) spectra and bispectra ([20], also see Appendix A) of pointed dg categories with weak equivalences and dualities. Invariance theorems of $\mathbf{GW}$-spectra and $\mathbb{G}W$-bispectra, similar to that of K-theory, were established in [20, Theorems 6.5, 8.9]. Contrary to K-theory and GW-theory, Balmer defined Witt theory for Triangulated categories with dualities [1], which encompasses the Derived categories with dualities. Therefore, the shifted Witt groups are invariant with respect to equivalences of derived categories [1, Theorem 6.2]. Another common thread among these three areas is the exactness properties of the associated triangulated categories. In particular, the renowned Gersten complexes in K-theory, GW-theory and Witt theory, are obtained by routine manipulation (see Remark 4.5) of the respective invariants, by such derived equivalences and exactness properties of the associated triangulated categories. For our purpose, some of the existing exactness theorems [2,4] of derived categories would suffice. Therefore, we first consider equivalences of certain derived categories, over quasi-projective schemes, which we state subsequently.

The readers are referred to Notations 2.1 for clarifications regarding notations and the definition of grade. Other than the notations explained above, for integers $k \geq 0$, $\mathbb{M}^k(X)$ will denote the category of coherent $\mathcal{O}_X$-modules $\mathcal{F}$ with finite locally free dimension, and $grade(\mathcal{F}) \geq k$. We prove that, for a noetherian quasi-projective scheme X over an affine scheme $\operatorname{Spec}(A)$, and integers $k \geq 0$, the functor of the derived categories

$$\zeta : \mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is an equivalence}$$

(see Theorem 3.1). We also prove that the functor of the derived categories

$$\beta : \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is faithfully full}$$

(see Theorem 3.2). Consequently, the functor $\mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right)$ is faithfully full. Combining the results in [13], we have the following summary of results. Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{D}^b(C\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\zeta} & \mathcal{D}^b(\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\iota} & \mathscr{D}^{k+1}(\mathbb{M}(X)) & \xleftarrow[\sim]{\iota'} & \mathscr{D}^{k+1}(\mathscr{V}(X)) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \eta \\
\mathcal{D}^b(C\mathbb{M}^{k}(X)) & \xrightarrow[\zeta]{\sim} & \mathcal{D}^b(\mathbb{M}^{k}(X)) & \xrightarrow[\iota]{\sim} & \mathscr{D}^{k}(\mathbb{M}(X)) & \xleftarrow[\iota']{\sim} & \mathscr{D}^{k}(\mathscr{V}(X))
\end{array} \qquad (1)$$