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The Grothendieck ring of varieties and of the theory of algebraically closed fields

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ABSTRACT

In each characteristic, there is a canonical homomorphism from the Grothendieck ring of varieties to the Grothendieck ring of sets definable in the theory of algebraically closed fields. We prove that this homomorphism is an isomorphism in characteristic zero. In positive characteristics, we exhibit specific elements in the kernel of the corresponding homomorphism of Grothendieck *semi*rings. The comparison of these two Grothendieck rings in positive characteristics seems to be an open question, related to the difficult problem of cancellativity of the Grothendieck semigroup of varieties.

1. Introduction

Of the many occurrences of Grothendieck rings in algebraic geometry, there are two closely related ones that are the subjects of this note. One is Grothendieck's original definition: the generators are isomorphism classes of varieties, and the relations stem from open-closed decompositions into subvarieties. See Bittner [1] for a careful discussion and presentation in terms of smooth varieties, and Looijenga [5] for how localizations and completions of this ring give rise to motivic measures. The other definition originates in geometric model theory, as an instance of the Grothendieck ring of models of a first-order theory. Here, in line with the general aims of model theory, the objects of study are formulas of first order logic, and the subsets of an ambient model they define. The natural notion of morphism becomes a definable map, and in the Grothendieck ring of definable sets, it is natural to permit as relations all definable decompositions. Let us specialize to the theory of algebraically closed fields. Thanks to the existence of "elimination of quantifiers" from first order formulas, definable sets coincide with the loci of points satisfying a boolean combination of polynomial equalities in affine space, i.e. constructible sets. A morphism between constructible sets is a point-map whose graph is constructible. In this approach, varieties are seen as 'point-clouds' rather than as ringed spaces,

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and morphisms need not be continuous. See Krajíček and Scanlon [4] for a very readable exposition of this Grothendieck ring and some of its uses in logic.

There is a natural homomorphism from the algebraic geometer's Grothendieck ring of varieties, denoted $K_0(\mathsf{var}_k)$ in this paper, to the model theorist's, that we denote $K_0(\mathsf{constr}_k)$. This homomorphism is an isomorphism when k is algebraically closed of characteristic zero. This has been known in the model theory community for quite some time, and (as the author has learned after this work was completed) it follows from Prop. 3.8, Cor. 3.11 and Prop. 3.13 of Nicaise–Sebag [9]. It may still be useful to give a direct proof of this result, using only basic properties of separated, quasi-finite morphisms. We actually prove the slightly stronger result that the canonical comparison map from the Grothendieck semiring of varieties to the Grothendieck semiring of algebraically closed fields, is an isomorphism in characteristic zero. (Recall that a semiring is a ring-like structure without the requirement that additive inverses exist. In the Grothendieck semiring of varieties resp. algebraically closed fields, any element can be represented as a formal linear combination of objects with positive integer coefficients; hence, ultimately, as a single variety resp. constructible set. The Grothendieck semiring determines the corresponding Grothendieck ring, but not conversely.)

In positive characteristics, the situation is subtle. Conceptually, the reason for the difference is the absence in positive characteristic of generic smoothness. That makes it difficult to 'spread out' information given on the level of closed points, which is what is available in $K_0(\mathsf{constr}_k)$, to open subsets, and hence make a conclusion about $K_0(\mathsf{var}_k)$ using Noetherian induction. We will prove that in positive characteristics, the canonical comparison map from the Grothendieck semiring $SK_0(\mathsf{var}_k)$ of varieties to those of constructible sets, $SK_0(\mathsf{constr}_k)$, is surjective but not injective. This leaves open the question whether $K_0(\mathsf{var}_k)$ and $K_0(\mathsf{constr}_k)$ are isomorphic in positive characteristics too. A resolution of this problem seems to require a better understanding of the canonical homomorphism $SK_0(\mathsf{var}_k) \to K_0(\mathsf{var}_k)$ in positive characteristics.

Let us give precise definitions. For an algebraically closed field k, let k-variety mean separated, reduced scheme of finite type over k. The Grothendieck semiring $SK_0(\mathsf{var}_k)$ is the commutative monoid (i.e. set with associative, commutative binary operation, with unit) generated by symbols $[X]$, one for each k-variety X, subject to the relations

- $[X] = [Y]$ if X and Y are isomorphic over k
- $[X] = [U] + [X - U]$ for any variety X with open subvariety U and closed complement $X - U$.

The product of k-varieties induces a commutative semiring structure on $SK_0(\mathsf{var}_k)$. The Grothendieck ring $K_0(\mathsf{var}_k)$ is defined analogously, based on the free abelian group generated by the symbols $[X]$.

A constructible subset of a scheme is one that can be written as a finite boolean combination of Zariski-closed subsets, considered as point-sets. Let constr_k be the category whose objects are pairs $\langle U, \mathbb{A}^n\rangle$ where U is a constructible subset of affine n-space $\mathbb{A}^n$ over k, and where a morphism $f : \langle U, \mathbb{A}^n\rangle \to \langle V, \mathbb{A}^m\rangle$ is a set-theoretic function $U \to V$ whose graph is a constructible subset of $\mathbb{A}^{n+m}$. The Grothendieck semiring $SK_0(\mathsf{constr}_k)$ of constructible sets is the commutative monoid generated by symbols $[\langle U, \mathbb{A}^n\rangle]$ corresponding to objects of constr_k, subject to the relations

- $[\langle U, \mathbb{A}^n\rangle] = [\langle V, \mathbb{A}^m\rangle]$ if $\langle U, \mathbb{A}^n\rangle$ and $\langle V, \mathbb{A}^m\rangle$ are isomorphic in constr_k
- $[\langle U, \mathbb{A}^n\rangle] = [\langle V, \mathbb{A}^n\rangle] + [\langle U - V, \mathbb{A}^n\rangle]$ whenever $V \subseteq U$.

For a scheme X, write $|X|$ for the set of points of its underlying topological space. Recall that there is a canonical (surjective) map $|X \times_k Y| \xrightarrow{p} |X| \times |Y|$. For constructible subsets $U \subseteq \mathbb{A}^n$, $V \subseteq \mathbb{A}^m$, $p^{-1}(U \times V)$ is a constructible subset of $\mathbb{A}^{n+m}$. This turns $SK_0(\mathsf{constr}_k)$ into a semiring. The Grothendieck ring $K_0(\mathsf{constr}_k)$ is the commutative ring defined by the same generators and relations.

Given a finite decomposition of a variety X into pairwise disjoint affine constructible sets $C_i \subseteq \mathbb{A}^{d_i}$, $i \in I$, define